REFLEXIVE FUNCTORS OF MODULES IN COMMUTATIVE ALGEBRA

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ABSTRACT. Reflexive functors of modules are ubiquitous in Algebraic Geometry, mainly in the theory of linear representations of group schemes, and in "duality theories". In this paper we study and determine reflexive functors of modules and we give many properties of reflexive functors of modules, of algebras and of bialgebras.

1. INTRODUCTION

Let $X = \operatorname{Spec} A$ be an affine scheme over a field K. We can regard X as a covariant functor of sets over the category of commutative K-algebras through its functor of points. Namely, let $X^{\cdot}$ be defined by $X^{\cdot}(S) = \operatorname{Hom}_{K-alg}(A, S)$. If $X = \operatorname{Spec} K[x_1, \dots, x_n]/(p_1, \dots, p_m)$ then

$$X^{\cdot}(S) := \{s \in S^n \colon p_1(s) = \cdots = p_m(s) = 0\}$$

By the Yoneda Lemma, $\operatorname{Hom}_{K-sch}(X, Y) = \operatorname{Hom}_{funct.}(X^{\cdot}, Y^{\cdot})$, and it is well known that X is an affine K-scheme of groups if and only if $X^{\cdot}$ is a functor of groups.

We can regard K as functor of rings $\mathcal{K}$, by defining $\mathcal{K}(S) := S$, for all commutative K-algebras S. Let V be a K-vector space. We can regard V as a covariant functor of $\mathcal{K}$-modules, $\mathcal{V}$, by defining $\mathcal{V}(S) := V \otimes_K S$. We will say that $\mathcal{V}$ is the $\mathcal{K}$-quasi-coherent module associated with V. If $V = \oplus_I K$ then $\mathcal{V}(S) = \oplus_I S$. It holds that the category of K-vector spaces, $\mathcal{C}_{\text{K-vect}}$, is equivalent to the category of quasi-coherent $\mathcal{K}$-modules, $\mathcal{C}_{\text{qs-coh }\mathcal{K}\text{-mod}}$: the functors $\mathcal{C}_{\text{K-vect}} \rightsquigarrow \mathcal{C}_{\text{qs-coh }\mathcal{K}\text{-mod}}$, $V \rightsquigarrow \mathcal{V}$ and $\mathcal{C}_{\text{qs-coh }\mathcal{K}\text{-mod}} \rightsquigarrow \mathcal{C}_{\text{K-vect}}$, $\mathcal{V} \rightsquigarrow \mathcal{V}(K)$ give the equivalence.

It is well known that the theory of linear representations of a group scheme $G = \operatorname{Spec} A$ can be developed, via their associated functors, as a theory of an abstract group and its linear representations. That is, the category of linear representations of the group scheme G is equivalent to the category of quasi-coherent $G^{\cdot}$-modules.

Given a functor of $\mathcal{K}$-modules, $\mathbb{M}$ (that is, a covariant functor from the category of commutative K-algebras to the category of abelian groups, with a structure of $\mathcal{K}$-module), we denote $\mathbb{M}^* := \mathbb{H}om_{\mathcal{K}}(\mathbb{M}, \mathcal{K})$. We say that $\mathbb{M}$ is a reflexive functor of modules if $\mathbb{M} = \mathbb{M}^{**}$.

Reflexive functors of modules are ubiquitous in Algebraic Geometry, mainly in the theory of linear representations of group schemes, and in "duality theories": Quasi-coherent modules are reflexive (even when K is a commutative ring, see [2]). Let $\mathbb{X}$ be a functor of sets and $\mathbb{A}_{\mathbb{X}} := \mathbb{H}om_{funct.}(\mathbb{X}, \mathcal{K})$. We say that $\mathbb{X}$ is an affine functor if $\mathbb{A}_{\mathbb{X}}$ is reflexive and $\mathbb{X} = \operatorname{Spec} \mathbb{A}_{\mathbb{X}} := \mathbb{H}om_{\mathcal{K}-alg}(\mathbb{A}_{\mathbb{X}}, \mathcal{K})$, see [8] for details

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(we warn the reader that in the literature affine functors are sometimes defined to be functors of points of affine schemes). In [8], we prove that affine schemes, formal schemes and the completion of an affine scheme along a closed set are affine functors. Let $\mathbb{G}$ be an affine functor of monoids. $\mathbb{A}^*_{\mathbb{G}}$ is a functor of algebras and the category of $\mathbb{G}$-modules is equivalent to the category of $\mathbb{A}^*_{\mathbb{G}}$-modules. Applications of these results include Cartier duality, neutral Tannakian duality for affine group schemes and the equivalence between formal groups and Lie algebras in characteristic zero (see [8]). In order to prove these results it is necessary to study and to determine reflexive functors of modules, algebras and bialgebras.

Some natural questions emerge: Is the family of reflexive functors a monster family? Is this family closed under tensor products? Is this family closed under homomorphisms?

In this paper we prove:

(1) Every reflexive functor of $\mathcal{K}$-modules is a functor of $\mathcal{K}$-submodules of a funtor of $\mathcal{K}$-modules $\prod_I \mathcal{K}$ (see 4.2).
(2) A functor of $\mathcal{K}$-modules is reflexive if and only if it is the inverse limit of its quasi-coherent quotients (see 4.4).
(3) If I is a totally ordered set and $\{f_{ij} \colon \mathcal{V}_i \to \mathcal{V}_j\}_{i\geq j\in I}$ is an inverse system of quasi-coherent $\mathcal{K}$-modules, then $\lim\limits_{\substack{\leftarrow \\ i\in I}} \mathcal{V}_i$ is a reflexive functor of $\mathcal{K}$-modules.
   We do not know if arbitrary inverse limits of quasi-coherent modules are reflexive, that is, if proquasi-coherent modules are reflexive. Then, we can not assure that reflexive functors have the same properties as the reflexive functors of $\mathfrak{F}$, defined below.
(4) If $\mathbb{M}$ and $\mathbb{M}'$ are reflexive functors of $\mathcal{K}$-modules, then $\mathrm{Hom}_{\mathcal{K}}(\mathbb{M},\mathbb{M}') \subseteq \mathrm{Hom}_K(\mathbb{M}(K),\mathbb{M}'(K))$ (see 3.11). If $\mathbb{A}$ is a reflexive functor and a functor of $\mathcal{K}$-algebras and $\mathbb{M},\mathbb{M}'$ are reflexive functors of $\mathbb{A}$-modules, then a morphism of $\mathcal{K}$-modules $\mathbb{M}\to\mathbb{M}'$ is a morphism of $\mathbb{A}$-modules if and only if $\mathbb{M}(K)\to\mathbb{M}'(K)$ is a morphism of $\mathbb{A}(K)$-modules. Let V be a vector space. If $\mathcal{V}$ is an $\mathbb{A}$-module, then the set of quasi-coherent $\mathbb{A}$-submodules of $\mathcal{V}$ is equal to the set of $\mathbb{A}(K)$-submodules of V (see 3.17 and 4.13).

Now assume $K = R$ is a commutative ring. In section 5, we define a wide family $\mathfrak{F}$ of reflexive functors of $\mathcal{R}$-modules satisfying:

(1) If M and N are free R-modules, then $\mathcal{M},\mathcal{M}^*,\mathbb{H}om_{\mathcal{R}}(\mathcal{M},\mathcal{N})\in\mathfrak{F}$.
(2) Every functor of $\mathcal{R}$-modules $\mathbb{M}\in\mathfrak{F}$ is proquasi-coherent.
(3) If $\mathbb{M},\mathbb{M}'\in\mathfrak{F}$, then $\mathbb{H}om_{\mathcal{R}}(\mathbb{M},\mathbb{M}')\in\mathfrak{F}$ and $(\mathbb{M}\otimes_{\mathcal{R}}\mathbb{M}')^{**}\in\mathfrak{F}$, which satisfies
$$\mathrm{Hom}_{\mathcal{R}}((\mathbb{M}\otimes_{\mathcal{R}}\mathbb{M}')^{**},\mathbb{M}'') = \mathrm{Hom}_{\mathcal{R}}(\mathbb{M}\otimes_{\mathcal{R}}\mathbb{M}',\mathbb{M}'')$$
   for every reflexive functor of $\mathcal{R}$-modules, $\mathbb{M}''$.
(4) If $\mathbb{A},\mathbb{B}\in\mathfrak{F}$ are functors of proquasi-coherent algebras, then $(\mathbb{A}^*\otimes_{\mathcal{R}}\mathbb{B}^*)^*\in\mathfrak{F}$ and it is a functor of proquasi-coherent algebras, which satisfies
$$\mathrm{Hom}_{\mathcal{R}-alg}((\mathbb{A}^*\otimes_{\mathcal{R}}\mathbb{B}^*)^*,\mathbb{C}) = \mathrm{Hom}_{\mathcal{R}-alg}(\mathbb{A}\otimes_{\mathcal{R}}\mathbb{B},\mathbb{C})$$
   for every functor of proquasi-coherent algebras, $\mathbb{C}$.
(5) The functor $\mathcal{C}_{\mathfrak{F}-bialg} \rightsquigarrow \mathcal{C}_{\mathfrak{F}-bialg}$, $\mathbb{B}\rightsquigarrow\mathbb{B}^*$ is a categorical anti-equivalence, where $\mathcal{C}_{\mathfrak{F}-bialg}$ is the category of functors of proquasi-coherent bialgebras (that is, $\mathbb{B}\in\mathcal{C}_{\mathfrak{F}-bialg}$ if $\mathbb{B}\in\mathfrak{F}$, $\mathbb{B}$ and $\mathbb{B}^*$ are functors of proquasi-coherent

algebras and the dual morphisms of the multiplication morphism and unit morphism of $\mathbb{B}^*$ are morphisms of functors of algebras).

Let A be a free R-module, then $\mathcal{A} \in \mathfrak{F}$. A is an R-bialgebra if and only if $\mathcal{A}$ is a functor of proquasi-coherent bialgebras (see Proposition 5.26). In the literature, there have been many attempts to obtain a well-behaved duality for non finite bialgebras (see [9] and references therein). One of them, for example, states that the functor that associates with each bialgebra A over a field K the so-called dual bialgebra A° is auto-adjoint ($A^\circ := \lim\limits_{\substack{\to \\ I\in J}} (A/I)^*$, where J is the set of bilateral ideals $I \subset A$ such that $\dim_K A/I < \infty$, see [1]). Another one associates with each bialgebra A over a pseudocompact ring R the bialgebra A^* endowed with a certain topology (see [4, Exposé VII_B 2.2.1]).

(6) If $\mathbb{M}, \mathbb{M}' \in \mathfrak{F}$, then $\mathrm{Hom}_{\mathcal{R}}(\mathbb{M}, \mathbb{M}') \subseteq \mathrm{Hom}_R(\mathbb{M}(R), \mathbb{M}'(R))$. If $\mathbb{A} \in \mathfrak{F}$ is a functor of $\mathcal{R}$-algebras and $\mathbb{M}, \mathbb{M}' \in \mathfrak{F}$ are functors of $\mathbb{A}$-modules, then a morphism of $\mathcal{R}$-modules $\mathbb{M} \to \mathbb{M}'$ is a morphism of $\mathbb{A}$-modules if and only if $\mathbb{M}(R) \to \mathbb{M}'(R)$ is a morphism of $\mathbb{A}(R)$-modules. Let M be an R-module. If $\mathcal{M}$ is an $\mathbb{A}$-module, then the set of quasi-coherent $\mathbb{A}$-submodules of $\mathcal{M}$ is equal to the set of $\mathbb{A}(R)$-submodules of M.

This paper completes [2] and it is essentially self contained.

2. Preliminaries

Let R be a commutative ring (associative with a unit). All functors considered in this paper are covariant functors over the category of commutative R-algebras (always assumed to be associative with a unit). A functor $\mathbb{X}$ is said to be a functor of sets (resp. monoids, etc.) if $\mathbb{X}$ is a functor from the category of commutative R-algebras to the category of sets (resp. monoids, etc.).

Notation 2.1. *For simplicity, given a functor of sets $\mathbb{X}$, we sometimes use $x \in \mathbb{X}$ to denote $x \in \mathbb{X}(S)$. Given $x \in \mathbb{X}(S)$ and a morphism of commutative R-algebras $S \to S'$, we still denote by x its image by the morphism $\mathbb{X}(S) \to \mathbb{X}(S')$.*

Let $\mathcal{R}$ be the functor of rings defined by $\mathcal{R}(S) := S$, for all commutative R-algebras S. A functor of sets $\mathbb{M}$ is said to be a functor of $\mathcal{R}$-modules if we have morphisms of functors of sets, $\mathbb{M} \times \mathbb{M} \to \mathbb{M}$ and $\mathcal{R} \times \mathbb{M} \to \mathbb{M}$, so that $\mathbb{M}(S)$ is an S-module, for every commutative R-algebra S. A functor of algebras (associative with a unit), $\mathbb{A}$, is said to be a functor of $\mathcal{R}$-algebras if we have a morphism of functors of algebras $\mathcal{R} \to \mathbb{A}$ (and $\mathcal{R}(S) = S$ commutes with all the elements of $\mathbb{A}(S)$, for every commutative R-algebra S).

Given a commutative R-algebra S, we denote by $\mathbb{M}_{|S}$ the functor $\mathbb{M}$ restricted to the category of commutative S-algebras.

Let $\mathbb{M}$ and $\mathbb{M}'$ be functors of $\mathcal{R}$-modules. A morphism of functors of $\mathcal{R}$-modules $f\colon \mathbb{M} \to \mathbb{M}'$ is a morphism of functors such that the defined morphisms $f_S\colon \mathbb{M}(S) \to \mathbb{M}'(S)$ are morphisms of S-modules, for all commutative R-algebras S. We will denote by $\mathrm{Hom}_{\mathcal{R}}(\mathbb{M}, \mathbb{M}')$ the set of all the morphisms of $\mathcal{R}$-modules from $\mathbb{M}$ to $\mathbb{M}'$. We will denote by $\mathbb{H}om_{\mathcal{R}}(\mathbb{M}, \mathbb{M}')$[1] the functor of $\mathcal{R}$-modules

$$\mathbb{H}om_{\mathcal{R}}(\mathbb{M}, \mathbb{M}')(S) := \mathrm{Hom}_{\mathcal{S}}(\mathbb{M}_{|S}, \mathbb{M}'_{|S})$$

[1] In this paper, we will only consider functors $\mathbb{M}$ and $\mathbb{M}'$ such that $\mathrm{Hom}_{\mathcal{S}}(\mathbb{M}_{|S}, \mathbb{M}'_{|S})$ are sets, for all S. In [2], in order for $\mathrm{Hom}_{\mathcal{R}}(\mathbb{M}, \mathbb{M}')$ to be a set instead of taking into account the category

Obviously,

$$(\mathbb{H}om_{\mathcal{R}}(\mathbb{M},\mathbb{M}'))_{|S} = \mathbb{H}om_{\mathcal{S}}(\mathbb{M}_{|S},\mathbb{M}'_{|S})$$

Notation 2.2. *We denote* $\mathbb{M}^* = \mathbb{H}om_{\mathcal{R}}(\mathbb{M},\mathcal{R})$.

Notation 2.3. *Tensor products, direct limits, inverse limits, kernels, cokernels, images, etc., of functors of $\mathcal{R}$-modules are regarded in the category of functors of $\mathcal{R}$-modules.*

Definition 2.4. *Given an R-module M, the functor of $\mathcal{R}$-modules $\mathcal{M}$ defined by $\mathcal{M}(S) := M\otimes_R S$ is called a quasi-coherent $\mathcal{R}$-module.*

Proposition 2.5. [2, 1.3] *For every functor of $\mathcal{R}$-modules $\mathbb{M}$ and every R-module M, it holds that*

$$\mathrm{Hom}_{\mathcal{R}}(\mathcal{M},\mathbb{M}) = \mathrm{Hom}_R(M,\mathbb{M}(R))$$

Proof. Given an $\mathcal{R}$-linear morphism $f:\mathcal{M}\to\mathbb{M}$, we have for every R-algebra S a morphism of S-modules $f_S : M\otimes_R S\to\mathbb{M}(S)$ and a commutative diagram

$$\begin{array}{ccc} M\otimes_R S & \xrightarrow{f_S} & \mathbb{M}(S) \\ \uparrow & & \uparrow \\ M & \xrightarrow{f_R} & \mathbb{M}(R) \end{array}$$

Hence, the morphism of S-modules f_S is determined by f_R. □

The functors $M \rightsquigarrow \mathcal{M}$, $\mathcal{M}\rightsquigarrow\mathcal{M}(R)=M$ establish an equivalence between the category of $\mathcal{R}$-modules and the category of quasi-coherent $\mathcal{R}$-modules ([2, 1.12]). In particular, $\mathrm{Hom}_{\mathcal{R}}(\mathcal{M},\mathcal{M}')=\mathrm{Hom}_R(M,M')$. For any pair of R-modules M and N, the quasi-coherent module associated with $M\otimes_R N$ is $\mathcal{M}\otimes_{\mathcal{R}}\mathcal{N}$. $\mathcal{M}_{|S}$ is the quasi-coherent $\mathcal{S}$-module associated with $M\otimes_R S$

The functor $\mathcal{M}^* = \mathbb{H}om_{\mathcal{R}}(\mathcal{M},\mathcal{R})$ is called an $\mathcal{R}$-module scheme. Moreover, $\mathcal{M}^*(S) = \mathrm{Hom}_S(M\otimes_R S,S) = \mathrm{Hom}_R(M,S)$ and it is easy to check that $(\mathcal{M}^*)_{|S}$ is an $\mathcal{S}$-module scheme.

Definition 2.6. *Given a commutative R-algebra A, let $(\mathrm{Spec}\,A)^{\cdot}$ be the functor defined by $(\mathrm{Spec}\,A)^{\cdot}(S) := \mathrm{Hom}_{\mathrm{R-alg}}(A,S)$, for each commutative R-algebra S. This functor will be called the functor of points of* $\mathrm{Spec}\,A$.

By Yoneda's lemma (see [7, Appendix A5.3]*),* $\mathrm{Hom}_{\mathrm{func}}((\mathrm{Spec}\,A)^{\cdot},\mathbb{M}) = \mathbb{M}(A)$.

Given an R-module M, we will denote by $S^{\cdot}_R M$ the symmetric algebra of M. Let us recall the next well-known lemma (see [5, II, §1, 2.1] or [4, Exposé VII_B, 1.2.4]).

Lemma 2.7. [2, 1.6] *If M is an R-module, then $\mathcal{M}^* = (\mathrm{Spec}\,S^{\cdot}_R M)^{\cdot}$ as functors of $\mathcal{R}$-modules.*

Proof. For every commutative R-algebra S, it holds that

$$\mathcal{M}^*(S) = \mathrm{Hom}_R(M,S) = \mathrm{Hom}_{\mathrm{R-alg}}(S^{\cdot}_R M,S) = (\mathrm{Spec}\,S^{\cdot}_R M)^{\cdot}(S)$$

□

Proposition 2.8. [2, 1.8] *Let M, M' be R-modules. Then*

$$\mathbb{H}om_{\mathcal{R}}(\mathcal{M}^*,\mathcal{M}') = \mathcal{M}\otimes_{\mathcal{R}}\mathcal{M}'$$

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of commutative R-algebras, we considered an infinite set I and the category of commutative R-algebras whose cardinal is less than or equal to $\mathrm{car}\,(I^{\mathbb{N}})$ (see [5, General conventions]).

Proof. We know that $\mathcal{M}^*$ is represented by $\operatorname{Spec} S^{\cdot}_R M$, therefore

$$\operatorname{Hom}_{\mathcal{R}}(\mathcal{M}^*, \mathcal{M}') \subseteq \operatorname{Hom}_{\text{func}}(\mathcal{M}^*, \mathcal{M}') = \mathcal{M}'(S^{\cdot}_R M) = S^{\cdot}_R M \otimes_R M'$$

However, in order for $w \in S^{\cdot}_R M \otimes_R M'$ to be a linear application, it must be $w \in M \otimes_R M'$. Hence, $\operatorname{Hom}_{\mathcal{R}}(\mathcal{M}^*, \mathcal{M}') = M \otimes_R M'$.

For every R-algebra S, we have that

$$\begin{aligned} \mathbb{H}om_{\mathcal{R}}(\mathcal{M}^*, \mathcal{M}')(S) &= \operatorname{Hom}_{\mathcal{S}}(\mathcal{M}^*{}_{|S}, \mathcal{M}'{}_{|S}) = \operatorname{Hom}_{\mathcal{S}}((\mathcal{M} \otimes_{\mathcal{R}} \mathcal{S})^*, \mathcal{M}' \otimes_{\mathcal{R}} \mathcal{S}) \\ &= (M \otimes_R S) \otimes_S (M' \otimes_R S) = (\mathcal{M} \otimes_{\mathcal{R}} \mathcal{M}')(S) \end{aligned}$$

□

As a corollary we obtain the following theorem.

Theorem 2.9. [2, 1.10] *Let M be an R-module. Then*

$$\mathcal{M}^{**} = \mathcal{M}$$

The functors $\mathcal{M} \rightsquigarrow \mathcal{M}^*$ and $\mathcal{M}^* \rightsquigarrow \mathcal{M}^{**} = \mathcal{M}$ establish an anti-equivalence between the categories of quasi-coherent modules and module schemes. An $\mathcal{R}$-module scheme $\mathcal{M}^*$ is a quasi-coherent $\mathcal{R}$-module if and only if M is a projective R-module of finite type (see [3]).

Let us recall the Formula of adjoint functors.

Notation 2.10. *Let $i^* \colon R \to S$ be a commutative R-algebra. Given a functor of $\mathcal{R}$-modules, $\mathbb{M}$, let $i^*\mathbb{M}$ be the functor of $\mathcal{S}$-modules defined by $(i^*\mathbb{M})(S') := \mathbb{M}(S')$. Given a functor of $\mathcal{S}$-modules, $\mathbb{N}$, let $i_*\mathbb{N}$ be the functor of $\mathcal{R}$-modules defined by $(i_*\mathbb{N})(R') := \mathbb{N}(S \otimes_R R')$.*

Formula of adjoint functors 2.11. [2, 1.12] *Let $\mathbb{M}$ be a functor of $\mathcal{R}$-modules and let $\mathbb{N}$ be a functor of $\mathcal{S}$-modules. Then, it holds that*

$$\operatorname{Hom}_{\mathcal{S}}(i^*\mathbb{M}, \mathbb{N}) = \operatorname{Hom}_{\mathcal{R}}(\mathbb{M}, i_*\mathbb{N})$$

Proof. Given a $w \in \operatorname{Hom}_{\mathcal{S}}(i^*\mathbb{M}, \mathbb{N})$, we have morphisms $w_{S\otimes R'} : \mathbb{M}(S \otimes R') \to \mathbb{N}(S \otimes R')$ for each commutative R-algebra R'. By composition with the morphisms $\mathbb{M}(R') \to \mathbb{M}(S \otimes R')$, we have the morphisms $\phi_{R'} : \mathbb{M}(R') \to \mathbb{N}(S \otimes R') = i_*\mathbb{N}(R')$, which in their turn define $\phi \in \operatorname{Hom}_{\mathcal{R}}(\mathbb{M}, i_*\mathbb{N})$.

Given a $\phi \in \operatorname{Hom}_{\mathcal{R}}(\mathbb{M}, i_*\mathbb{N})$, we have morphisms $\phi_{S'} : \mathbb{M}(S') \to i_*\mathbb{N}(S') = \mathbb{N}(S \otimes S')$ for each S-algebra S'. By composition with the morphisms $\mathbb{N}(S \otimes S') \to \mathbb{N}(S')$, we have the morphisms $w_{S'} : \mathbb{M}(S') \to \mathbb{N}(S')$, which in their turn define $w \in \operatorname{Hom}_{\mathcal{S}}(i^*\mathbb{M}, \mathbb{N})$.

Now we shall show that $w \mapsto \phi$ and $\phi \mapsto w$ are mutually inverse. Given $w \in \operatorname{Hom}_{\mathcal{S}}(i^*\mathbb{M}, \mathbb{N})$ we have $\phi \in \operatorname{Hom}_{\mathcal{R}}(\mathbb{M}, i_*\mathbb{N})$. Let us prove that the latter defines w again. We have the following diagram, where S' is a commutative S-algebra and i, p the obvious morphisms,

$$\begin{tikzcd}
\mathbb{M}(S') \arrow[r, "i"] \arrow[dr, equal] & \mathbb{M}(S \otimes S') \arrow[r, "w_{S\otimes S'}"] \arrow[d] & \mathbb{N}(S \otimes S') \arrow[d, "p"] \\
& \mathbb{M}(S') \arrow[r, "w_{S'}"] & \mathbb{N}(S')
\end{tikzcd}$$

The composite morphism $p \circ w_{S\otimes S'} \circ i = p \circ \phi_{S'}$ is that assigned to ϕ, and coincides with $w_{S'}$ since the whole diagram is commutative.

Given $\phi \in \operatorname{Hom}_{\mathcal{R}}(\mathbb{M}, i_*\mathbb{N})$ we have $w \in \operatorname{Hom}_{\mathcal{S}}(i^*\mathbb{M}, \mathbb{N})$. Let us see that the latter defines ϕ. We have the following diagram, where R' is a commutative R-algebra and r, j, p the obvious morphisms,

$$\begin{tikzcd}
\mathbb{M}(R') \arrow[r, "r"] \arrow[d, "\phi_{R'}"] & \mathbb{M}(S \otimes R') \arrow[r, "w_{S\otimes R'}"] \arrow[d, "\phi_{S\otimes R'}"] & \mathbb{N}(S \otimes R') \\
(i_*\mathbb{N})(R') \arrow[r, "j"] & (i_*\mathbb{N})(S \otimes R') \arrow[r, equal] & \mathbb{N}(S \otimes S \otimes R') \arrow[u, "p"]
\end{tikzcd}$$

The composite morphism $w_{S\otimes R'} \circ r$ assigned to w agrees with $\phi_{R'}$, since $p \circ j = Id$ and the whole diagram is commutative.

$\square$

Corollary 2.12. *Let $\mathbb{M}$ be a functor of $\mathcal{R}$-modules. Then*

$$\mathbb{M}^*(S) = \operatorname{Hom}_{\mathcal{R}}(\mathbb{M}, \mathcal{S})$$

for all commutative R-algebras S.

Proof. $\mathbb{M}^*(S) = \operatorname{Hom}_{\mathcal{S}}(\mathbb{M}_{|S}, \mathcal{S}) \overset{2.11}{=} \operatorname{Hom}_{\mathcal{R}}(\mathbb{M}, \mathcal{S})$. $\square$

Definition 2.13. *Let $\mathbb{M}$ be a functor of $\mathcal{R}$-modules. We will say that $\mathbb{M}^*$ is a dual functor. We will say that a functor of $\mathcal{R}$-modules $\mathbb{M}$ is reflexive if $\mathbb{M} = \mathbb{M}^{**}$.*

Examples 2.14. *Quasi-coherent modules and module schemes are reflexive functors of $\mathcal{R}$-modules.*

Proposition 2.15. *Let $\mathbb{M}$ be a functor of $\mathcal{R}$-modules such that $\mathbb{M}^*$ is a reflexive functor. The closure of dual functors of $\mathcal{R}$-modules of $\mathbb{M}$ is $\mathbb{M}^{**}$, that is, it holds the functorial equality*

$$\operatorname{Hom}_{\mathcal{R}}(\mathbb{M}, \mathbb{M}') = \operatorname{Hom}_{\mathcal{R}}(\mathbb{M}^{**}, \mathbb{M}')$$

for every dual functor of $\mathcal{R}$-modules $\mathbb{M}'$.

Proof. Write $\mathbb{M}' = \mathbb{N}^*$. Then, $\operatorname{Hom}_{\mathcal{R}}(\mathbb{M}, \mathbb{M}') = \operatorname{Hom}_{\mathcal{R}}(\mathbb{M} \otimes \mathbb{N}, \mathcal{R}) = \operatorname{Hom}_{\mathcal{R}}(\mathbb{N}, \mathbb{M}^*) = \operatorname{Hom}_{\mathcal{R}}(\mathbb{N} \otimes \mathbb{M}^{**}, \mathcal{R}) = \operatorname{Hom}_{\mathcal{R}}(\mathbb{M}^{**}, \mathbb{M}')$. $\square$

Proposition 2.16. *Let $\mathbb{A}$ be a functor of $\mathcal{R}$-algebras such that $\mathbb{A}^*$ is a reflexive functor of $\mathcal{R}$-modules. The closure of dual functors of $\mathcal{R}$-algebras of $\mathbb{A}$ is $\mathbb{A}^{**}$, that is, it holds the functorial equality*

$$\operatorname{Hom}_{\mathcal{R}-alg}(\mathbb{A}, \mathbb{B}) = \operatorname{Hom}_{\mathcal{R}-alg}(\mathbb{A}^{**}, \mathbb{B})$$

for every dual functor of $\mathcal{R}$-algebras $\mathbb{B}$.

Moreover, endowing a dual functor of $\mathcal{R}$-modules $\mathbb{M}^*$ with a structure of $\mathbb{A}$-module is equivalent to endowing $\mathbb{M}^*$ with a structure of $\mathbb{A}^{**}$-module.

Proof. Given a dual functor of $\mathcal{R}$-modules $\mathbb{M}^*$, by induction on n

$$\begin{aligned}
\mathbb{H}om_{\mathcal{R}}(\mathbb{A} \otimes \overset{n}{\cdots} \otimes \mathbb{A}, \mathbb{M}^*) &= \mathbb{H}om_{\mathcal{R}}(\mathbb{A} \otimes \overset{n-1}{\cdots} \otimes \mathbb{A}, \mathbb{H}om_{\mathcal{R}}(\mathbb{A}, \mathbb{M}^*)) \\
&\overset{2.15}{=} \mathbb{H}om_{\mathcal{R}}(\mathbb{A} \otimes \overset{n-1}{\cdots} \otimes \mathbb{A}, \mathbb{H}om_{\mathcal{R}}(\mathbb{A}^{**}, \mathbb{M}^*)) \\
&\overset{\text{Ind.Hyp.}}{=} \mathbb{H}om_{\mathcal{R}}(\mathbb{A}^{**} \otimes \overset{n-1}{\cdots} \otimes \mathbb{A}^{**}, \mathbb{H}om_{\mathcal{R}}(\mathbb{A}^{**}, \mathbb{M}^*)) \\
&= \mathbb{H}om_{\mathcal{R}}(\mathbb{A}^{**} \otimes \ldots \otimes \mathbb{A}^{**}, \mathbb{M}^*).
\end{aligned}$$

Let $i\colon \mathbb{A} \to \mathbb{A}^{**}$ be the natural morphism. The multiplication morphism $m\colon \mathbb{A}\otimes\mathbb{A} \to \mathbb{A}$ defines a unique morphism $m'\colon \mathbb{A}^{**}\otimes\mathbb{A}^{**} \to \mathbb{A}^{**}$ such that the diagram

$$\begin{tikzcd} \mathbb{A}\otimes\mathbb{A} \arrow[r, "i\otimes i"] \arrow[d, "m"] & \mathbb{A}^{**}\otimes\mathbb{A}^{**} \arrow[d, "m'"] \\ \mathbb{A} \arrow[r, "i"] & \mathbb{A}^{**} \end{tikzcd}$$

is commutative, because $\mathrm{Hom}_{\mathcal{R}}(\mathbb{A}\otimes\mathbb{A}, \mathbb{A}^{**}) = \mathrm{Hom}_{\mathcal{R}}(\mathbb{A}^{**}\otimes\mathbb{A}^{**}, \mathbb{A}^{**})$. It follows easily that the algebra structure of $\mathbb{A}$ defines an algebra structure on $\mathbb{A}^{**}$. Let us only check that m' satisfies the associative property: The morphisms $m'\circ(m'\otimes\mathrm{Id})$, $m'\circ(\mathrm{Id}\otimes m')\colon \mathbb{A}^{**}\otimes\mathbb{A}^{**}\otimes\mathbb{A}^{**} \to \mathbb{A}^{**}$ are equal because

$$\begin{aligned}(m'\circ(m'\otimes\mathrm{Id}))\circ(i\otimes i\otimes i) &= m'\circ(i\otimes i)\circ(m\otimes\mathrm{Id}) = i\circ m\circ(m\otimes\mathrm{Id})\\ &= i\circ m\circ(\mathrm{Id}\otimes m) = m'\circ(i\otimes i)\circ(\mathrm{Id}\otimes m) = (m'\circ(\mathrm{Id}\otimes m'))\circ(i\otimes i\otimes i)\end{aligned}$$

The kernel of the morphism $\mathrm{Hom}_{\mathcal{R}}(\mathbb{A},\mathbb{B}) \to \mathrm{Hom}_{\mathcal{R}}(\mathbb{A}\otimes_{\mathcal{R}}\mathbb{A},\mathbb{B})$, $f\mapsto f\circ m - m\circ(f\otimes f)$, coincides kernel of the morphism $\mathrm{Hom}_{\mathcal{R}}(\mathbb{A}^{**},\mathbb{B}) \to \mathrm{Hom}_{\mathcal{R}}(\mathbb{A}^{**}\otimes_{\mathcal{R}}\mathbb{A}^{**},\mathbb{B})$, $f\mapsto f\circ m' - m\circ(f\otimes f)$. Then, $\mathrm{Hom}_{\mathcal{R}-alg}(\mathbb{A},\mathbb{B}) = \mathrm{Hom}_{\mathcal{R}-alg}(\mathbb{A}^{**},\mathbb{B})$.

Finally, given a dual functor of $\mathcal{R}$-modules $\mathbb{M}^*$, then $\mathbb{E}nd_{\mathcal{R}}\mathbb{M}^* = (\mathbb{M}^*\otimes\mathbb{M})^*$ is a dual functor of $\mathcal{R}$-algebras and

$$\mathrm{Hom}_{\mathcal{R}-alg}(\mathbb{A}, \mathbb{E}nd_{\mathcal{R}}\mathbb{M}^*) = \mathrm{Hom}_{\mathcal{R}-alg}(\mathbb{A}^{**}, \mathbb{E}nd_{\mathcal{R}}\mathbb{M}^*)$$

Hence, endowing $\mathbb{M}^*$ with a structure of $\mathbb{A}$-module is equivalent to endowing it with a structure of $\mathbb{A}^{**}$-module

□

Example 2.17. *Let $G = \mathrm{Spec}\,A$ a K-group scheme and $\mathcal{K}[G^\cdot]$ the functor defined by $\mathcal{K}[G^\cdot](S) = \{$formal sums $s_1g_1+\cdots+s_ng_n$, $n\in\mathbb{N}$, $s_i\in S$ and $g_i\in G^\cdot(S)\}$. In [3] we prove that $\mathcal{K}[G^\cdot]^* = \mathcal{A}$ and $\mathcal{K}[G^\cdot]^{**} = \mathcal{A}^*$, then the category of (rational) G-modules is equivalent to the category of quasi-coherent $\mathcal{K}[G^\cdot]$-modules, which is equivalent to the category of quasi-coherent $\mathcal{A}^*$-modules.*

3. Functors of modules with the D property

Notation 3.1. *Let us denote $\mathbb{M}(\mathcal{R})$ the quasi-coherent module associated with the R-module $\mathbb{M}(R)$, that is, $\mathbb{M}(\mathcal{R})(S) := \mathbb{M}(R)\otimes_R S$.*

There exists a natural morphism $\mathbb{M}(\mathcal{R}) \to \mathbb{M}$, $m\otimes s\mapsto s\cdot m$.

Definition 3.2. *We will say that a functor of $\mathcal{R}$-modules $\mathbb{M}$ holds the D property if the natural morphism $\mathbb{M}^* \to \mathbb{M}(\mathcal{R})^*$ is injective.*

Example 3.3. *Quasi-coherent modules hold the D property, because $\mathcal{M}(\mathcal{R}) = \mathcal{M}$.*

Example 3.4. *If $M = \oplus_I R$ is a free R-module, then $\mathcal{M}^*$ holds the D property: Consider the obvious morphisms $\oplus_I\mathcal{R} \to \mathcal{M}^*(\mathcal{R}) \to \mathcal{M}^*$. Dually, the composite morphism $\mathcal{M} = \mathcal{M}^{**} \to \mathcal{M}^*(\mathcal{R})^* \to \prod_I\mathcal{R}$ is injective, then $\mathcal{M}^{**} \to \mathcal{M}^*(\mathcal{R})^*$ is injective.*

Note 3.5. *The direct limit of a direct system of functors of modules with the D property holds the D property. Every quotient of a functor of $\mathcal{R}$-modules with the D property holds the D property.*

Theorem 3.6. *A functor of $\mathcal{R}$-modules $\mathbb{M}$ holds the D property if and only if for every R-module N the map*

$$\mathrm{Hom}_{\mathcal{R}}(\mathbb{M}, \mathcal{N}) \to \mathrm{Hom}_R(\mathbb{M}(R), N), \quad f \mapsto f_R$$

is injective.

Proof. If the natural morphism $\mathbb{M}^* \to \mathbb{M}(\mathcal{R})^*$ is injective, then $\mathrm{Hom}_{\mathcal{R}}(\mathbb{M}, \mathcal{S}) \subseteq \mathrm{Hom}_R(\mathbb{M}(R), S)$ for all commutative R-algebras S. Given an R-module N, consider the R-algebra $S := R \oplus N$, with the multiplication operation $(r, n) \cdot (r', n') := (rr', rn' + r'n)$. Then,

$$\mathrm{Hom}_{\mathcal{R}}(\mathbb{M}, \mathcal{R} \oplus \mathcal{N}) = \mathrm{Hom}_{\mathcal{R}}(\mathbb{M}, \mathcal{S}) \subseteq \mathrm{Hom}_R(\mathbb{M}(R), S) = \mathrm{Hom}_R(\mathbb{M}(R), R \oplus N)$$

Hence, $\mathrm{Hom}_{\mathcal{R}}(\mathbb{M}, \mathcal{N}) \subseteq \mathrm{Hom}_R(\mathbb{M}(R), N)$

Reciprocally, $\mathbb{M}^*(S) = \mathrm{Hom}_{\mathcal{R}}(\mathbb{M}, \mathcal{S}) \hookrightarrow \mathrm{Hom}_R(\mathbb{M}(R), S) = \mathbb{M}(\mathcal{R})^*(S)$ for all commutative R-algebras S, and $\mathbb{M}^* \hookrightarrow \mathbb{M}(\mathcal{R})^*$. □

Corollary 3.7. *Let $R = K$ be a field. A functor of $\mathcal{K}$-modules $\mathbb{M}$ holds the D property if and only if the natural morphism $\mathbb{M}^*(K) \to \mathbb{M}(K)^* := \mathrm{Hom}_K(\mathbb{M}(K), K)$ is injective.*

Proof. We only have to prove the sufficiency. Let $N = \oplus_I K$ be a K-vector space. The diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{K}}(\mathbb{M}, \mathcal{N}) \arrow[rrrr] \arrow[d, hook] & & & & \mathrm{Hom}_K(\mathbb{M}(K), N) \arrow[d, hook] \\
\mathrm{Hom}_{\mathcal{K}}(\mathbb{M}, \prod^I \mathcal{K}) = \prod^I \mathrm{Hom}_{\mathcal{K}}(\mathbb{M}, \mathcal{K}) \arrow[r, hook] & \prod^I \mathrm{Hom}_K(\mathbb{M}(K), K) = \mathrm{Hom}_K(\mathbb{M}(K), \prod^I K) & & &
\end{tikzcd}$$

is commutative. Then, the morphism $\mathrm{Hom}_{\mathcal{K}}(\mathbb{M}, \mathcal{N}) \to \mathrm{Hom}_K(\mathbb{M}(K), N)$ is injective and $\mathbb{M}$ holds the D property. □

Proposition 3.8. *Property D is stable by base change. That is, if $\mathbb{M}$ is a functor of $\mathcal{R}$-modules with the D property and S is a commutative R-algebra, then the functor of $\mathcal{S}$-modules $\mathbb{M}_{|S}$ holds the D property.*

Proof. Let S be a commutative R-algebra and let N be an S-module. The diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{S}}(\mathbb{M}_{|S}, \mathcal{N}) \arrow[r, equal, "2.11"] \arrow[d] & \mathrm{Hom}_{\mathcal{R}}(\mathbb{M}, \mathcal{N}) \arrow[d, hook, "3.6"] \\
\mathrm{Hom}_{\mathcal{S}}(\mathbb{M}(S), N) \arrow[r] & \mathrm{Hom}_R(\mathbb{M}(R), N)
\end{tikzcd}$$

is commutative, then the morphism $\mathrm{Hom}_{\mathcal{S}}(\mathbb{M}_{|S}, \mathcal{N}) \to \mathrm{Hom}_{\mathcal{S}}(\mathbb{M}(S), N)$ is injective and $\mathbb{M}_{|S}$ holds the D property. □

Definition 3.9. *A functor of modules $\mathbb{M}$ is said to be (linearly) separated if for each commutative R-algebra S and $m \in \mathbb{M}(S)$ there exist a commutative S-algebra T and a $w \in \mathbb{M}^*(T)$ such that $w(m) \neq 0$ (that is, the natural morphism $\mathbb{M} \to \mathbb{M}^{**}$, $m \mapsto \tilde{m}$, where $\tilde{m}(w) := w(m)$ for all $w \in \mathbb{M}^*$, is injective).*

Every subfunctor of modules of a separated functor of modules is separated.

Example 3.10. *If $\mathbb{M}$ is a dual functor of modules, then it is separated: Given $0 \neq w \in \mathbb{M} = \mathbb{N}^*$, there exists an $n \in \mathbb{N}$ such that $w(n) \neq 0$. Let $\tilde{n} \in \mathbb{M}^*$ be defined by $\tilde{n}(w') := w'(n)$, for all $w' \in \mathbb{M}$. Then $\tilde{n}(w) \neq 0$.*

Theorem 3.11. *Let $\mathbb{M}$ be a functor of $\mathcal{R}$-modules. $\mathbb{M}$ holds the D property if and only if the morphism*

$$\mathrm{Hom}_{\mathcal{R}}(\mathbb{M}, \mathbb{M}') \to \mathrm{Hom}_R(\mathbb{M}(R), \mathbb{M}'(R)), \quad f \mapsto f_R$$

is injective, for all separated $\mathcal{R}$-modules, $\mathbb{M}'$ (such that $\mathbb{M}'^*$ are well defined).

Proof. By Theorem 3.6, we only have to prove the necessity. The morphism $\mathrm{Hom}_{\mathcal{R}}(\mathbb{M}, \mathbb{M}') \to \mathrm{Hom}_{\mathcal{R}}(\mathbb{M}'^*, \mathbb{M}^*)$, $f \mapsto f^*$ is injective: If $f \neq 0$ there exists an $m \in \mathbb{M}$ such that $f(m) \neq 0$. Then, there exists a $w' \in \mathbb{M}'^*$ such that $0 \neq w'(f(m)) = f^*(w')(m)$. Therefore, $f^*(w') \neq 0$ and $f^* \neq 0$.

From the diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{R}}(\mathbb{M}, \mathbb{M}') \arrow[r, hook] \arrow[d] & \mathrm{Hom}_{\mathcal{R}}(\mathbb{M}'^*, \mathbb{M}^*) \arrow[r, hook, "3.2"] & \mathrm{Hom}_{\mathcal{R}}(\mathbb{M}'^*, \mathbb{M}(\mathcal{R})^*) \\
\mathrm{Hom}_R(\mathbb{M}(R), \mathbb{M}'(R)) \arrow[r, equal] & \mathrm{Hom}_{\mathcal{R}}(\mathbb{M}(\mathcal{R}), \mathbb{M}') \arrow[ur] &
\end{tikzcd}$$

one deduces that the morphism $\mathrm{Hom}_{\mathcal{R}}(\mathbb{M}, \mathbb{M}') \to \mathrm{Hom}_R(\mathbb{M}(R), \mathbb{M}'(R))$ is injective. □

Corollary 3.12. *Let $R = K$ be a field and let $\mathbb{M}$, $\mathbb{M}'$ be functors of $\mathcal{K}$-modules with the D property, then $\mathbb{M} \otimes_{\mathcal{K}} \mathbb{M}'$ holds the D property.*

Proof. It is due to the inclusion $(\mathbb{M} \otimes_{\mathcal{K}} \mathbb{M}')^*(K) = \mathrm{Hom}_{\mathcal{K}}(\mathbb{M} \otimes \mathbb{M}', \mathcal{K}) = \mathrm{Hom}_{\mathcal{K}}(\mathbb{M}, \mathbb{M}'^*)$ $\overset{3.11}{\hookrightarrow} \mathrm{Hom}_K(\mathbb{M}(K), \mathbb{M}'^*(K)) \hookrightarrow \mathrm{Hom}_K(\mathbb{M}(K), \mathbb{M}'(K)^*) = \mathrm{Hom}_K(\mathbb{M}(K) \otimes \mathbb{M}'(K), K)$ $= (\mathbb{M} \otimes_{\mathcal{K}} \mathbb{M}')(K)^*$. □

Lemma 3.13. *A functor of $\mathcal{R}$-modules $\mathbb{M}$ holds the D property if and only if the cokernel of every $\mathcal{R}$-module morphism from $\mathbb{M}$ to a quasi-coherent module is quasi-coherent, that is, the cokernel of any morphism $f\colon \mathbb{M} \to \mathcal{N}$ is the quasi-coherent module associated with* $\mathrm{Coker}\, f_R$.

Proof. ⇒) Let $f\colon \mathbb{M} \to \mathcal{N}$ be a morphism of $\mathcal{R}$-modules. Let $N' = \mathrm{Coker}\, f_R$ and let $\pi\colon \mathcal{N} \to \mathcal{N}'$ be the natural epimorphism. As $(\pi \circ f)_R = 0$, $\pi \circ f = 0$ by Theorem 3.6. Then, $\mathrm{Coker}\, f = \mathcal{N}'$.

⇐) Let $f\colon \mathbb{M} \to \mathcal{N}$ be a morphism of $\mathcal{R}$-modules. If $f_R = 0$ then $\mathrm{Coker}\, f = \mathcal{N}$ and $f = 0$. Therefore, $\mathbb{M}$ holds the D property, by Theorem 3.6. □

Note 3.14. *If $R = K$ is a field, the kernel of every morphism between quasi-coherent modules is quasi-coherent. Then, $\mathbb{M}$ holds the D property if and only if the image of every morphism from $\mathbb{M}$ to a quasi-coherent module is quasi-coherent.*

Theorem 3.15. *Let $R = K$ be a field and let $\mathbb{M}$ be a functor of $\mathcal{K}$-modules with the D property. Let $\{\mathcal{M}_i\}_{i \in I}$ be the set of the quasi-coherent quotients of $\mathbb{M}$. Then,*

$$\mathbb{M}^* = \lim_{\substack{\to \\ i \in I}} \mathcal{M}_i^*$$

Proof. Let S be a commutative K-algebra. $\mathbb{M}^*(S) = \mathrm{Hom}_{\mathcal{K}}(\mathbb{M}, \mathcal{S})$, by Corollary 2.12. The morphism $\lim\limits_{\substack{\rightarrow \\ i\in I}} \mathcal{M}_i^*(S) \to \mathrm{Hom}_{\mathcal{K}}(\mathbb{M}, \mathcal{S}) = \mathbb{M}^*(S)$ is obviously injective, and it is surjective by Lemma 3.13 and Note 3.14. Hence, $\mathbb{M}^* = \lim\limits_{\substack{\rightarrow \\ i\in I}} \mathcal{M}_i^*$. □

Corollary 3.16. *Let $R = K$ be a field. If $\mathbb{M}$ is a functor of $\mathcal{K}$-modules with the D property, then $\mathbb{M}^*$ holds the D property.*

Proof. It is a consequence of Theorem 3.15, Example 3.4 and Note 3.5. □

Proposition 3.17. *Let $\mathbb{A}$ be a functor of $\mathcal{K}$-algebras with the D property, let $\mathcal{M}$ and $\mathcal{N}$ be functors of $\mathbb{A}$-modules and let $M' \subset M$ be a K-vector subspace. Then,*

1. *$\mathcal{M}'$ is a quasi-coherent $\mathbb{A}$-submodule of $\mathcal{M}$ if and only if M' is an $\mathbb{A}(K)$-submodule of M.*
2. *A morphism $f\colon \mathcal{M} \to \mathcal{N}$ of functors of $\mathcal{K}$-modules is a morphism of $\mathbb{A}$-modules if and only if $f_K\colon M \to N$ is a morphism of $\mathbb{A}(K)$-modules.*

Proof. (1) Obviously, if $\mathcal{M}'$ is an $\mathbb{A}$-submodule of $\mathcal{M}$ then M' is an $\mathbb{A}(K)$-submodule of M. Inversely, let us assume M' is an $\mathbb{A}(K)$-submodule of M and let us consider the natural morphism of multiplication $\mathbb{A} \otimes_{\mathcal{K}} \mathcal{M}' \to \mathcal{M}$. The morphisms $\mathbb{A} \to \mathcal{M}$, $a \mapsto a \cdot m'$, for each $m' \in M'$, factors via $\mathcal{M}'$, then $\mathbb{A} \otimes_{\mathcal{K}} \mathcal{M}' \to \mathcal{M}$ factors via $\mathcal{M}'$. Therefore, $\mathcal{M}'$ is a functor of $\mathbb{A}$-submodules of $\mathcal{M}$.

(2) The morphism f is a morphism of $\mathbb{A}$-modules if and only if $F\colon \mathbb{A} \otimes \mathcal{M} \to \mathcal{N}$, $F(a \otimes m) := f(am) - af(m)$ is the zero morphism. Likewise, f_K is a morphism of $\mathbb{A}(K)$-modules if and only if $F_K\colon \mathbb{A}(K) \otimes M \to N$, $F_K(a \otimes m) = f_K(am) - af_K(m)$ is the zero morphism. Now, the proposition is a consequence of the inclusions,

$$\begin{aligned}\mathrm{Hom}_{\mathcal{K}}(\mathbb{A} \otimes \mathcal{M}, \mathcal{N}) = \mathrm{Hom}_{\mathcal{K}}(\mathbb{A}, \mathbb{H}om_{\mathcal{K}}(\mathcal{M}, \mathcal{N})) &\underset{3.11}{\subseteq} \mathrm{Hom}_K(\mathbb{A}(K), \mathrm{Hom}_{\mathcal{K}}(\mathcal{M}, \mathcal{N})) \\ &\subseteq \mathrm{Hom}_K(\mathbb{A}(K), \mathrm{Hom}_K(M, N)) = \mathrm{Hom}_K(\mathbb{A}(K) \otimes M, N)\end{aligned}$$

□

Proposition 3.18. *Let $\mathbb{A}$ be a functor of $\mathcal{K}$-algebras with the D property and let B be a K-algebra. Every morphism of $\mathcal{K}$-algebras $\phi\colon \mathbb{A} \to \mathcal{B}$ uniquely factors through an epimorphism of functors of algebras onto the quasi-coherent algebra associated with $\mathrm{Im}\,\phi_K$.*

Proof. By Note 3.14, the morphism $\phi\colon \mathbb{A} \to \mathcal{B}$ uniquely factors through an epimorphism $\phi'\colon \mathbb{A} \to \mathcal{B}'$, where $B' := \mathrm{Im}\,\phi_K$. Obviously B' is a K-subalgebra of B and ϕ' is a morphism of functors of algebras. □

4. Proquasi-coherent modules

Definition 4.1. *A functor of $\mathcal{R}$-modules is said to be a proquasi-coherent module if it is an inverse limit of quasi-coherent modules.*

In this section, $R = K$ will be a field.

Proposition 4.2. *Let $R = K$ be a field and let $\mathbb{M}$ be a $\mathcal{K}$-module such that $\mathbb{M}^*$ is well defined. $\mathbb{M}$ is separated if and only if the morphism $\mathbb{M} \to \tilde{\mathbb{M}} := (\mathbb{M}^*(\mathcal{K}))^*$ is injective. Therefore, $\mathbb{M}$ is separated if and only if it is a $\mathcal{K}$-submodule of a $\mathcal{K}$-module scheme.*

Proof. Assume $\mathbb{M}$ is separated. Let $m \in \mathbb{M}(S)$ be such that $m = 0$ in $\bar{\mathbb{M}}(S)$. $\bar{\mathbb{M}}(S) = \mathbb{M}^*(\mathcal{K})^*(S)$=$\mathrm{Hom}_K(\mathbb{M}^*(K), S)$, then $m(w) := w(m) = 0$ for all $w \in \mathbb{M}^*(K)$.

Given a commutative S-algebra T, if one writes $T = \oplus_{i\in I} K \cdot e_i$, one notices that

$$\mathbb{M}^*(T) \overset{2.12}{=} \mathrm{Hom}_{\mathcal{K}}(\mathbb{M}, \mathcal{T}) = \mathrm{Hom}_{\mathcal{K}}(\mathbb{M}, \oplus_I \mathcal{K}) \subset \prod_I \mathrm{Hom}_{\mathcal{K}}(\mathbb{M}, \mathcal{K})$$

which assigns to every $w_T \in \mathbb{M}^*(T)$ a $(w_i) \in \prod \mathbb{M}^*(K)$. Explicitly, given $m' \in \mathbb{M}(T)$, then $w_T(m') = \sum_i w_i(m') \cdot e_i$. Therefore, $w_T(m) = 0$ for all $w_T \in \mathbb{M}^*(T)$. As $\mathbb{M}$ is separated, this means that $m = 0$, i.e., the morphism $\mathbb{M} \to \bar{\mathbb{M}}$ is injective.

Now, assume $\mathbb{M} \to \bar{\mathbb{M}}$ is injective. Observe that $\bar{\mathbb{M}}$ is separated because is reflexive. Then $\mathbb{M}$ is separated.

Finally, the second statement of the proposition is obvious. □

Proposition 4.3. *If $\mathbb{M}$ is a proquasi-coherent $\mathcal{K}$-module then it is a dual $\mathcal{K}$-module and it is a direct limit of $\mathcal{K}$-shemes of modules. In particular, proquasi-coherent modules hold the D property.*

Proof. $\mathbb{M} = \lim\limits_{\leftarrow} \mathcal{M}_i = (\lim\limits_{\rightarrow} \mathcal{M}_i^*)^*$. As $\lim\limits_{\rightarrow} \mathcal{M}_i^*$ holds the D property, its dual, which is $\mathbb{M}$, is a direct limit of $\mathcal{K}$-module schemes, by Theorem 3.15. □

Theorem 4.4. *Let $R = K$ be a field. $\mathbb{M}$ is a reflexive functor of $\mathcal{K}$-modules if and only if $\mathbb{M}$ is equal to the inverse limit of its quasi-coherent quotients. In particular, reflexive functors of $\mathcal{K}$-modules are proquasi-coherent and hold the D property.*

Proof. Suppose that $\mathbb{M}$ is reflexive. $\mathbb{M}^*$ is separated, because it is a dual functor of modules. By Proposition 4.2, the morphism $\mathbb{M}^* \to \mathbb{M}(\mathcal{K})^*$ is injective. Then, $\mathbb{M}$ holds the D property. Let $\{\mathcal{M}_i\}_{i\in I}$ be the set of the quasi-coherent quotients of $\mathbb{M}$. Then, $\mathbb{M}^* = \lim\limits_{\substack{\rightarrow \\ i\in I}} \mathcal{M}_i^*$, by Theorem 3.15. Therefore, $\mathbb{M} = \mathbb{M}^{**} = \lim\limits_{\substack{\leftarrow \\ i\in I}} \mathcal{M}_i$.

Suppose now that $\mathbb{M}$ is equal to the inverse limit of its quasi-coherent quotients. By Proposition 4.3, $\mathbb{M}$ holds the D property. By Theorem 3.15, $\mathbb{M} = \mathbb{M}^{**}$.

□

Let $R = \mathbb{Z}$ and $M = \mathbb{Z}/2\mathbb{Z}$. Then, $\mathbb{M} := \mathcal{M}^*$ is reflexive but it does not hold D property, because $\mathbb{M}(\mathcal{R})^* = 0$, since $\mathbb{M}(R) = 0$.

Proposition 4.5. *Let $f\colon \mathbb{P} \to \mathbb{M}$ be a morphism of functors of $\mathcal{K}$-modules. If $\mathbb{P}$ is proquasi-coherent and $\mathbb{M}$ is separated then $\mathbb{K}er\, f$ is proquasi-coherent.*

Proof. Let V be a K-vector space such that there exists an injective morphism $\mathbb{M} \hookrightarrow \mathcal{V}^*$. We can assume $\mathbb{M} = \mathcal{V}^* = \prod_I \mathcal{K}$. Given $I' \subset I$ let $f_{I'}$ the composition of f with the obvious projection $\prod_I \mathcal{K} \to \prod_{I'} \mathcal{K}$. Then

$$\mathbb{K}er\, f = \lim_{\substack{\leftarrow \\ I'\subset I,\, \#I'<\infty}} \mathbb{K}er\, f_{I'}$$

It is sufficient to prove that $\mathbb{K}er\, f_{I'}$ is proquasi-coherent, since the inverse limit of proquasi-coherent modules is proquasi-coherent. As $\#I' < \infty$ it is sufficient to prove that the kernel of every morphism $f\colon \mathbb{P} \to \mathcal{K}$ is proquasi-coherent.

If $f\colon \mathbb{P} \to \mathcal{K}$ is the zero morphism the proposition is obvious. Assume $f \neq 0$. Then, f is surjective. Let us write $\mathbb{P} = \lim\limits_{\substack{\leftarrow \\ i}} \mathcal{V}_i$ and let $v = (v_i) \in \lim\limits_{\substack{\leftarrow \\ i}} V_i = \mathbb{P}(K)$

be a vector such that $f_K((v_i)) \neq 0$. Then $\mathbb{P} = \mathbb{K}er\, f \oplus \mathcal{K} \cdot v$. Let $\bar{V}_i := V_i/\langle v_i \rangle$. Let us prove that $\mathbb{K}er\, f = \lim\limits_{\overleftarrow{i}} \bar{\mathcal{V}}_i$: Let i' be such that $v_{i'} \neq 0$. Consider the exact sequences

$$0 \to \mathcal{K} \cdot v_i \to \mathcal{V}_i \to \bar{\mathcal{V}}_i \to 0, \qquad (i > i')$$

Dually, we have the exact sequences

$$0 \to \bar{\mathcal{V}}_i^* \to \mathcal{V}_i^* \to \mathcal{K} \to 0$$

Taking the direct limit we have the split exact sequence

$$0 \to \lim_{\overrightarrow{i}} (\bar{\mathcal{V}}_i^*) \to \lim_{\overrightarrow{i}} (\mathcal{V}_i^*) \to \mathcal{K} \to 0$$

Dually, we have the exact sequence

$$0 \to \mathcal{K} \cdot v \to \mathbb{P} \to \lim_{\overleftarrow{i}} \bar{\mathcal{V}}_i \to 0$$

Then, $\mathbb{K}er\, f \to \lim\limits_{\overleftarrow{i}} \bar{\mathcal{V}}_i$, $(v_i)_i \mapsto (\bar{v}_i)_i$ is an isomorphism.

□

Corollary 4.6. *Every direct summand of a proquasi-coherent module is proquasi-coherent.*

Theorem 4.7. *Let $\mathbb{M}$ be a functor of $\mathcal{K}$-modules. $\mathbb{M}$ is proquasi-coherent if and only if $\mathbb{M}$ is a dual functor of $\mathcal{K}$-modules and it holds the D property.*

Proof. By Proposition 4.3, we only have to prove the sufficiency. Let us write $\mathbb{M} = \mathbb{N}^*$. The dual morphism of the natural morphism $\mathbb{N} \to \mathbb{N}^{**}$ is a retraction of the natural morphism $\mathbb{M} \to \mathbb{M}^{**}$. Then, $\mathbb{M}^{**} = \mathbb{M} \oplus \mathbb{M}'$. $\mathbb{M}$ is proquasi-coherent, because $\mathbb{M}^{**}$ is proquasi-coherent, by Theorem 3.15.

□

Corollary 4.8. *A functor of $\mathcal{K}$-modules is proquasi-coherent if and only if it is the dual functor of $\mathcal{K}$-modules of a functor of $\mathcal{K}$-modules with the D property.*

Proof. If $\mathbb{M} = \lim\limits_{\overleftarrow{i}} \mathcal{M}_i$ is proquasi-coherent, then $\mathbb{M} = (\lim\limits_{\overrightarrow{i}} \mathcal{M}_i^*)^*$, and $\lim\limits_{\overrightarrow{i}} \mathcal{M}_i^*$ holds the D property. If $\mathbb{M}'$ holds the D property, then $\mathbb{M}'^*$ holds the D property, by Corollary 3.16. By Theorem 4.7, $\mathbb{M}'^*$ is proquasi-coherent. □

Proposition 4.9. *Let M be an R-module. Then,*

$$\mathbb{H}om_{\mathcal{R}}(\prod_I \mathcal{R}, \mathcal{M}) = \oplus_I \mathbb{H}om_{\mathcal{R}}(\mathcal{R}, \mathcal{M}) = \oplus_I \mathcal{M}$$

Proof. $\mathbb{H}om_{\mathcal{R}}(\prod_I \mathcal{R}, \mathcal{M}) = \mathbb{H}om_{\mathcal{R}}((\oplus_I \mathcal{R})^*, \mathcal{M}) \overset{2.8}{=} (\oplus_I \mathcal{R}) \otimes \mathcal{M} = \oplus_I \mathcal{M}$. □

Proposition 4.10. *Let I be a totally ordered set and $\{f_{ij} \colon M_i \to M_j\}_{i \geq j \in I}$ an inverse system of K-modules. Then, $\lim\limits_{\overleftarrow{i}} \mathcal{M}_i$ is reflexive.*

Proof. $\lim\limits_{\leftarrow i} \mathcal{M}_i$ is a direct limit of submodule schemes $\mathcal{V}_j^*$, by 3.15 and 4.8. If all the vector spaces V_j are finite dimensional then $\lim\limits_{\leftarrow i} \mathcal{M}_i$ is quasi-coherent, then it is reflexive. In other case, there exists an injective morphism $f\colon \prod_{\mathbb{N}} \mathcal{K} \hookrightarrow \lim\limits_{\leftarrow i} \mathcal{M}_i$. Let $\pi_j\colon \lim\limits_{\leftarrow i} \mathcal{M}_i \to \mathcal{M}_j$ be the natural morphisms. Let $g_r\colon \mathcal{K}^r \hookrightarrow \prod_{\mathbb{N}} \mathcal{K}$ be defined by $g_r(\lambda_1, \cdots, \lambda_r) := (\lambda_1, \cdots, \lambda_r, 0, \cdots, 0, \cdots)$. Let $i_1 \in I$ be such that $\pi_{i_1} \circ f \circ g_1$ is injective. Recursively, let $i_n > i_{n-1}$ be such that $\pi_{i_n} \circ f \circ g_n$ is injective. If there exists a $j > i_n$ for all n, the composite morphism $\oplus_{\mathbb{N}} \mathcal{K} \subset \prod_{\mathbb{N}} \mathcal{K} \to \mathcal{M}_j$ is injective, and by Proposition 4.9 the morphism $\prod_{\mathbb{N}} \mathcal{K} \to \mathcal{M}_j$ factors through the projection onto a $\mathcal{K}^r$, which is contradictory. In conclusion, $\lim\limits_{\leftarrow i} \mathcal{M}_i = \lim\limits_{\leftarrow n\in\mathbb{N}} \mathcal{M}_{i_n}$.

Let $\mathcal{M}'_{i_r}$ be the image of $\lim\limits_{\leftarrow n} \mathcal{M}_{i_n}$ in $\mathcal{M}_{i_r}$. Then, $\lim\limits_{\leftarrow n} \mathcal{M}'_{i_n} = \lim\limits_{\leftarrow n} \mathcal{M}_{i_n}$. Let $H_n := \mathrm{Ker}[M'_{i_n} \to M'_{i_{n-1}}]$. Then, $\lim\limits_{\leftarrow n} \mathcal{M}_{i_n} \simeq \prod_n \mathcal{H}_n$. By Lemma 5.1, $\lim\limits_{\leftarrow n} \mathcal{M}_{i_n}$ is reflexive.

□

Note 4.11. *We do not know if every proquasi-coherent functor of $\mathcal{K}$-modules is reflexive.*

Proposition 4.12. *If $\mathbb{P}, \mathbb{P}'$ are proquasi-coherent $\mathcal{K}$-modules, then $\mathbb{H}om_{\mathcal{K}}(\mathbb{P}, \mathbb{P}')$ is proquasi-coherent. In particular, $\mathbb{P}^*$ and $(\mathbb{P} \otimes \mathbb{P}')^*$ are proquasi-coherent.*

Proof. Let us write $\mathbb{P} = \lim\limits_{\rightarrow i} \mathcal{V}_i^*$ and $\mathbb{P}' = \lim\limits_{\leftarrow j} \mathcal{V}'_j$. Then,

$$\mathbb{H}om_{\mathcal{K}}(\mathbb{P}, \mathbb{P}') = \mathbb{H}om_{\mathcal{K}}(\lim_{\rightarrow i} \mathcal{V}_i^*, \lim_{\leftarrow j} \mathcal{V}'_j) = \lim_{\leftarrow i,j} \mathbb{H}om_{\mathcal{K}}(\mathcal{V}_i^*, \mathcal{V}'_j) = \lim_{\leftarrow i,j} (\mathcal{V}_i \otimes \mathcal{V}'_j)$$

Hence, $\mathbb{H}om(\mathbb{P}, \mathbb{P}')$ is proquasi-coherent.

□

Proposition 4.13. *Let $\mathbb{A}$ be a functor of $\mathcal{K}$-algebras with the D property and let $\mathbb{P}, \mathbb{P}'$ be proquasi-coherent $\mathcal{K}$-modules and $\mathbb{A}$-modules. Then, a morphism of $\mathcal{K}$-modules, $f\colon \mathbb{P} \to \mathbb{P}'$, is a morphism of $\mathbb{A}$-modules if and only if $f_K\colon \mathbb{P}(K) \to \mathbb{P}'(K)$ is a morphism of $\mathbb{A}(K)$-modules.*

Proof. Proceed as in the proof of Proposition 3.17 (2).

□

5. A family $\mathfrak{F}$ of reflexive functors of $\mathcal{R}$-modules

Consider $\prod_{j\in J} \mathcal{R}$ as a functor of $\mathcal{R}$-algebras $((\lambda_i)_i \cdot (\mu_i)_i := (\lambda_i \cdot \mu_i)_i)$. If $\{\mathbb{M}_j\}_{j\in J}$ is a set of $\mathcal{R}$-modules, then $\oplus_{j\in J} \mathbb{M}_j$ and $\prod_{j\in J} \mathbb{M}_j$ are naturally functors of $\prod_{j\in J} \mathcal{R}$-modules.

Lemma 5.1. *Let $\mathbb{M}$ be a dual functor of $\mathcal{R}$-modules. If there exist a set of reflexive functors of $\mathcal{R}$-modules, $\{\mathbb{M}_j\}_{j\in J}$, and inclusions of $\prod_{j\in J} \mathcal{R}$-modules*

$$\oplus_{j\in J} \mathbb{M}_j \subseteq \mathbb{M} \subseteq \prod_{j\in J} \mathbb{M}_j$$

then

(1) $\mathbb{M}$ *is a reflexive functor of* $\mathcal{R}$*-modules.*
(2) *For every* R*-module* N *we have*

$$\oplus_{j\in J}\mathbb{H}om_{\mathcal{R}}(\mathbb{M}_j,\mathcal{N})\subseteq \mathbb{H}om_{\mathcal{R}}(\mathbb{M},\mathcal{N})\subseteq \prod_{j\in J}\mathbb{H}om_{\mathcal{R}}(\mathbb{M}_j,\mathcal{N})$$

Proof. Given $w\in \mathbb{H}om_{\mathcal{R}}(\mathbb{M},\mathcal{N})$, let $w_j := w_{|\mathbb{M}_j}$, for all $j\in J$. Given $m=(m_j)_j\in \mathbb{M}\subset \prod_j \mathbb{M}_j$, then $w_j(m_j)=0$ for all $j\in J$, except for a finite subset $I\subset J$, and $w(m)=\sum_{i\in I} w_i(m_i)$: Let $W\colon \prod_j \mathcal{R}\to\mathcal{N}$ be defined by $W((\lambda_j)_j):=w((\lambda_j m_j)_j)$. By Proposition 4.9, there exists a finite subset $I\subset J$ such that $W((\lambda_j)_j)=W((\lambda_i)_{i\in I})$. Hence, $w_j(m_j)=w(m_j)=0$ for all $j\in J\backslash I$, and $w(m)=w((m_j)_j)=w((m_i)_{i\in I})=\sum_{i\in I} w_i(m_i)$.

Then, $\oplus_{j\in J}\mathbb{H}om_{\mathcal{R}}(\mathbb{M}_j,\mathcal{N})\subseteq \mathbb{H}om_{\mathcal{R}}(\mathbb{M},\mathcal{N})\subseteq \prod_{j\in J}\mathbb{H}om_{\mathcal{R}}(\mathbb{M}_j,\mathcal{N})$.

In particular, we have

$$\oplus_j \mathbb{M}_j^*\subseteq \mathbb{M}^*\subseteq \prod_j \mathbb{M}_j^*$$

$\mathbb{M}^*$ is a $\prod_j \mathcal{R}$-module and again

$$\oplus_j \mathbb{M}_j\subseteq \mathbb{M}^{**}\subseteq \prod_j \mathbb{M}_j$$

We have $\oplus_j\mathbb{M}_j\subseteq\mathbb{M}\subseteq\mathbb{M}^{**}\subseteq\prod_j\mathbb{M}_j$, and again $(\mathbb{M}^{**})^*\subseteq\mathbb{M}^*\subseteq\prod_j\mathbb{M}_j^*$. The natural morphism $(\mathbb{M}^{**})^*\to\mathbb{M}^*$ is an epimorphism, because the natural morphism $\mathbb{M}^*\to(\mathbb{M}^*)^{**}$ is a section. Therefore, $\mathbb{M}^*=\mathbb{M}^{***}$.

The inclusion $\mathbb{M}\subseteq\mathbb{M}^{**}$ has a retraction, because $\mathbb{M}=\mathbb{M}'^*$ is a dual functor and the natural morphism $\mathbb{M}'^*\to(\mathbb{M}'^*)^{**}$ has a retraction. Then, $\mathbb{M}^{**}=\mathbb{M}\oplus\mathbb{M}''$. Dually, $\mathbb{M}^*=\mathbb{M}^{***}$, so $\mathbb{M}''^*=0$. Hence, $\mathbb{M}''=0$, because $\mathbb{M}''\subset\prod_j\mathbb{M}_j$ and for any $0\neq(m_j)\in\prod_j\mathbb{M}_j$ there exist a $j\in J$ and a $w_j\in\mathbb{M}_j^*$ such that $w_j(m_j)\neq 0$ (recall that the modules $\mathbb{M}_j$ are reflexive functors). Therefore, $\mathbb{M}=\mathbb{M}^{**}$.

□

Definition 5.2. *Let* $\mathfrak{F}$ *be the family of dual functors of* $\mathcal{R}$*-modules,* $\mathbb{M}$*, such that there exist a set* J *(which depends on* $\mathbb{M}$*), a structure of functor of* $\prod_J\mathcal{R}$*-modules on* $\mathbb{M}$ *and inclusions of functors of* $\prod_J\mathcal{R}$*-modules*

$$\oplus_J\mathcal{R}\subseteq\mathbb{M}\subseteq\prod_J\mathcal{R}$$

Note 5.3. *Every* $\mathbb{M}\in\mathfrak{F}$ *is reflexive, by Lemma 5.1.*

Examples 5.4. *If* V *is a free* $\mathcal{R}$*-module,* $\mathcal{V},\mathcal{V}^*\in\mathfrak{F}$*. If we have a set* $\{\mathbb{M}_i\in\mathfrak{F}\}_{i\in I}$*, then* $\oplus_{i\in I}\mathbb{M}_i,\prod_{i\in I}\mathbb{M}_i\in\mathfrak{F}$*, as it is easy to check.*

Let $V=\oplus_I R$ *and* $W=\oplus_J R$ *be free* R*-modules, then* $\mathbb{H}om_{\mathcal{R}}(\mathcal{V},\mathcal{W})\in\mathfrak{F}$*: we have the obvious inclusions of functors of* $\prod_{I\times J}\mathcal{R}$*-modules,* $\oplus_{I\times J}\mathcal{R}\subseteq\mathbb{H}om_{\mathcal{R}}(\mathcal{V},\mathcal{W})=\prod_I(\oplus_J\mathcal{R})\subseteq\prod_{I\times J}\mathcal{R}$*. This example has motivated Definition 5.2.*

Note 5.5. *A quasi-coherent module* $\mathcal{M}\in\mathfrak{F}$ *if and only if* M *is a free* R*-module: Let us write* $\oplus_J\mathcal{R}\subseteq\mathcal{M}\subseteq\prod_J\mathcal{R}$*. Let* $N=M/\oplus_J R$ *and let* $\pi\colon M\to N$ *be the quotient morphism. By Lemma 5.1, the morphism* $\mathrm{Hom}_R(M,N)\to\mathrm{Hom}_R(\oplus_J R,N)$ *is injective. As* $\pi\mapsto 0$*, then* $\pi=0$ *and* $M=\oplus_J R$*.*

Let $\mathfrak{F}'$ be the family of functors of $\mathcal{R}$-modules, $\mathbb{M}$, such that $\mathbb{M}$ is a direct summand of some functor of $\mathcal{R}$-modules of $\mathfrak{F}$. Obviously, $\mathcal{M}\in\mathfrak{F}'$ if M is a projective R-module. All other results obtained in this section for $\mathfrak{F}$ remain valid for $\mathfrak{F}'$.

Proposition 5.6. *If $\mathbb{M}'$ is a reflexive functor of $\mathcal{R}$-modules and $\mathbb{M}\in\mathfrak{F}$, then $\mathbb{H}om_{\mathcal{R}}(\mathbb{M}',\mathbb{M})$ is reflexive.*

Proof. $\mathbb{H}om_{\mathcal{R}}(\mathbb{M}',\mathbb{M})=(\mathbb{M}'\otimes\mathbb{M}^*)^*$ is a dual functor. Following previous notations, since $\oplus_J\mathcal{R}\subseteq\mathbb{M}\subseteq\prod_J\mathcal{R}$, we have the inclusions

$$(1)\quad \oplus_J\,\mathbb{M}'^*\subseteq\mathbb{H}om_{\mathcal{R}}(\mathbb{M}',\oplus_J\mathcal{R})\subseteq\mathbb{H}om_{\mathcal{R}}(\mathbb{M}',\mathbb{M})\subseteq\mathbb{H}om_{\mathcal{R}}(\mathbb{M}',\prod_J\mathcal{R})=\prod_J\mathbb{M}'^*$$

Therefore, by Lemma 5.1, $\mathbb{H}om_{\mathcal{R}}(\mathbb{M}',\mathbb{M})$ is reflexive. □

Proposition 5.7. *If $\mathbb{M}'$ is a reflexive functor of $\mathcal{R}$-modules and $\mathbb{M}\in\mathfrak{F}$, then $\mathbb{H}om_{\mathcal{R}}(\mathbb{M},\mathbb{M}')$ is reflexive.*

Proof. $\mathbb{H}om_{\mathcal{R}}(\mathbb{M},\mathbb{M}')=(\mathbb{M}\otimes\mathbb{M}'^*)^*$ is a dual functor. Let $0\neq f\in\mathbb{H}om_{\mathcal{R}}(\mathbb{M},\mathbb{M}')$. There exists $w\in\mathbb{M}'^*$ such that $w\circ f\neq 0$. Let us follow previous notations. By Lemma 5.1, $(w\circ f)_{|\oplus_J\mathcal{R}}\neq 0$, then $f_{|\oplus_J\mathcal{R}}\neq 0$. Therefore, $\mathbb{H}om_{\mathcal{R}}(\oplus_J\mathcal{R},\mathbb{M}')\supseteq\mathbb{H}om_{\mathcal{R}}(\mathbb{M},\mathbb{M}')$ and, likewise, $\mathbb{H}om_{\mathcal{R}}(\oplus_J\mathcal{R},\mathbb{M}')\supseteq\mathbb{H}om_{\mathcal{R}}(\prod_J\mathcal{R},\mathbb{M}')$. Hence, we have the inclusions

$$\prod_J\mathbb{M}'=\mathbb{H}om_{\mathcal{R}}(\oplus_J\mathcal{R},\mathbb{M}')\supseteq\mathbb{H}om_{\mathcal{R}}(\mathbb{M},\mathbb{M}')\supseteq\mathbb{H}om_{\mathcal{R}}(\prod_J\mathcal{R},\mathbb{M}')\supseteq\oplus_J\mathbb{M}'$$

Therefore, by Lemma 5.1, $\mathbb{H}om_{\mathcal{R}}(\mathbb{M},\mathbb{M}')$ is reflexive. □

Theorem 5.8. *If $\mathbb{M}',\mathbb{M}\in\mathfrak{F}$, then $\mathbb{H}om_{\mathcal{R}}(\mathbb{M}',\mathbb{M})\in\mathfrak{F}$.*

Proof. Let us write $\oplus_I\mathcal{R}\subseteq\mathbb{M}'\subseteq\prod_I\mathcal{R}$ and $\oplus_J\mathcal{R}\subseteq\mathbb{M}\subseteq\prod_J\mathcal{R}$. By Lemma 5.1, $\oplus_I\mathcal{R}\subseteq\mathbb{M}'^*\subseteq\prod_I\mathcal{R}$, then by Equation 1,

$$(2)\qquad \oplus_{I\times J}\,\mathcal{R}\subseteq\oplus_J\mathbb{M}'^*\subseteq\mathbb{H}om_{\mathcal{R}}(\mathbb{M}',\mathbb{M})\subseteq\prod_J\mathbb{M}'^*\subseteq\prod_{I\times J}\mathcal{R}$$

Observe that

$$(\prod_I\mathcal{R}\otimes\prod_J\mathcal{R})^*=\mathbb{H}om_{\mathcal{R}}(\prod_I\mathcal{R}\otimes\prod_J\mathcal{R},\mathcal{R})=\mathbb{H}om_{\mathcal{R}}(\prod_I\mathcal{R},\oplus_J\mathcal{R})\overset{2.8}{=}(\oplus_I\mathcal{R})\otimes(\oplus_J\mathcal{R})$$

$\mathbb{H}om_{\mathcal{R}}(\mathbb{M}',\mathbb{M})=(\mathbb{M}'\otimes\mathbb{M}^*)^*$ is a $\prod_I\mathcal{R}\otimes\prod_J\mathcal{R}$-module, then by Proposition 2.16, it is a $(\prod_I\mathcal{R}\otimes_{\mathcal{R}}\prod_J\mathcal{R})^{**}=(\oplus_I\mathcal{R}\otimes_{\mathcal{R}}\oplus_J\mathcal{R})^*=\prod_{I\times J}\mathcal{R}$-module. Finally, by Equation 2, $\mathbb{H}om_{\mathcal{R}}(\mathbb{M}',\mathbb{M})\in\mathfrak{F}$. □

Theorem 5.9. *If $\mathbb{M},\mathbb{M}'\in\mathfrak{F}$, then $(\mathbb{M}'\otimes\mathbb{M})^{**}\in\mathfrak{F}$ and it is the closure of dual functors of $\mathbb{M}'\otimes\mathbb{M}$.*

Proof. As $\mathbb{M}^*\in\mathfrak{F}$, we have $(\mathbb{M}'\otimes\mathbb{M})^*=\mathbb{H}om_{\mathcal{R}}(\mathbb{M}',\mathbb{M}^*)\in\mathfrak{F}$. Hence, firstly $(\mathbb{M}'\otimes\mathbb{M})^*$ is reflexive and by Proposition 2.15 the closure of dual functors of $\mathbb{M}'\otimes\mathbb{M}$ is $(\mathbb{M}'\otimes\mathbb{M})^{**}$, secondly $(\mathbb{M}'\otimes\mathbb{M})^{**}\in\mathfrak{F}$. □

Proposition 5.10. *Let $R=K$ be a field. Let I be a totally ordered set and $\{f_{ij}\colon M_i\to M_j\}_{i\geq j\in I}$ be an inverse system of morphisms of K-modules. Then, $\lim\limits_{\substack{\leftarrow\\ i\in I}}\mathcal{M}_i\in\mathfrak{F}$.*

Proof. $\lim\limits_{\substack{\leftarrow \\ i\in I}} \mathcal{M}_i$ is a direct product of $\mathcal{K}$-quasi-coherent modules, by the proof of Proposition 4.10. Hence, $\lim\limits_{\substack{\leftarrow \\ i\in I}} \mathcal{M}_i \in \mathfrak{F}$. □

Proposition 5.11. *If $\mathbb{M} \in \mathfrak{F}$, then $\mathbb{M}$ holds the D property.*

Proof. We have to prove that the morphism

$$\mathrm{Hom}_{\mathcal{R}}(\mathbb{M}, \mathcal{N}) \subseteq \mathrm{Hom}_R(\mathbb{M}(R), N),\ \ w \mapsto w_R$$

is injective, for all R-modules N. Let us follow previous notations. By Lemma 5.1, $w \in \mathrm{Hom}_{\mathcal{R}}(\mathbb{M}, \mathcal{N})$ is determined by $w_{|\oplus_J \mathcal{R}}$, and this one is determined by $(w_{|\oplus_J \mathcal{R}})_R$. As $\oplus_J R \subseteq \mathbb{M}(R)$, w is determined by w_R. □

Lemma 5.12. *Let $\mathbb{M} \in \mathfrak{F}$ and let N be an R-module. Then, every morphism of $\mathcal{R}$-modules $\phi\colon \mathbb{M} \to \mathcal{N}$ uniquely factors through an epimorphism onto the quasi-coherent module associated with the R-submodule of N, $\mathrm{Im}\,\phi_R \subseteq N$.*

Proof. Let us follows previous notations. Consider $\oplus_J \mathcal{R} \subseteq \mathbb{M} \subseteq \prod_J \mathcal{R}$. By Lemma 5.1, $\mathbb{M}^* \subseteq (\oplus_J \mathcal{R})^* = \prod_J \mathcal{R}$. The morphism $\phi_{|\oplus_J \mathcal{R}}\colon \oplus_J \mathcal{R} \to \mathcal{N}$ factors via the quasi-coherent module associated with $N' := \mathrm{Im}(\phi_{|\oplus_J \mathcal{R}})_R$. Then, the dual morphism $\phi^*\colon \mathcal{N}^* \to \mathbb{M}^* \subseteq \prod_J \mathcal{R}$, factors via, $\mathcal{N}'^*$. Hence, ϕ factors via a morphism $\phi'\colon \mathbb{M} \to \mathcal{N}'$. In particular, ϕ' is an epimorphism and $N' = \mathrm{Im}\,\phi_R$.

Uniqueness: Assume ϕ factors through an epimorphism $\phi''\colon \mathbb{M} \to \mathcal{N}''$ onto the quasi-coherent module associated with $N'' \subseteq N$. Observe that $N'' = \mathrm{Im}\,\phi_R = N'$. The morphisms ϕ, ϕ', ϕ'' are determined by $\phi_R, {\phi'}_R, {\phi''}_R$. Then, $\phi' = \phi''$, because ${\phi'}_R = {\phi''}_R$.

□

Theorem 5.13. *Let $\mathbb{M} \in \mathfrak{F}$. Let $\{\mathcal{M}_i\}_{i\in I}$ be the set of the quasi-coherent quotients of $\mathbb{M}$. Then, $\mathbb{M}^* = \lim\limits_{\substack{\rightarrow \\ i\in I}} \mathcal{M}_i^*$. Therefore,*

$$\mathbb{M} = \lim_{\substack{\leftarrow \\ i\in I}} \mathcal{M}_i.$$

Proof. Proceed as in the proof of 3.15 □

Proposition 5.14. *Let $\mathbb{A} \in \mathfrak{F}$ be a functor of $\mathcal{R}$-algebras and let $\mathbb{M}, \mathbb{M}' \in \mathfrak{F}$ be functors of $\mathbb{A}$-modules. Then, a morphism of $\mathcal{R}$-modules, $f\colon \mathbb{M} \to \mathbb{M}'$, is a morphism of $\mathbb{A}$-modules if and only if $f_R\colon \mathbb{M}(R) \to \mathbb{M}'(R)$ is a morphism of $\mathbb{A}(R)$-modules.*

Proof. Proceed as in Proposition 4.13.

□

Notation 5.15. *Let M an R-module and $M' \subseteq M$ an R-submodule. By abuse of notation we will say that $\mathcal{M}'$ is a quasi-coherent submodule of $\mathcal{M}$.*

Proposition 5.16. *Let $\mathbb{A} \in \mathfrak{F}$ be a functor of $\mathcal{R}$-algebras, let $\mathcal{M}$ be an $\mathbb{A}$-module and let $M' \subset M$ be an R-submodule. Then, $\mathcal{M}'$ is a quasi-coherent $\mathbb{A}$-submodule of $\mathcal{M}$ if and only if M' is an $\mathbb{A}(R)$-submodule of M.*

Proof. Obviously, if $\mathcal{M}'$ is an $\mathbb{A}$-submodule of $\mathcal{M}$ then M' is an $\mathbb{A}(R)$-submodule of M. Inversely, let us assume M' is an $\mathbb{A}(R)$-submodule of M and let us consider the natural morphism of multiplication $\mathbb{A}\otimes_{\mathcal{R}}\mathcal{M}'\to\mathcal{M}$. By Lemma 5.12, the morphisms $\mathbb{A}\to\mathcal{M}$, $a\mapsto a\cdot m'$, for each $m'\in M'$, uniquely factors via $\mathcal{M}'$, then $\mathbb{A}\otimes_{\mathcal{R}}\mathcal{M}'\to\mathcal{M}$ factors via $\mathcal{M}'$.

Let i be the morphism $\mathcal{M}'\to\mathcal{M}$. $F\colon\mathbb{A}\otimes_{\mathcal{R}}\mathbb{A}\to\mathcal{M}'$, $F(a\otimes a'):=a(a'm')-(aa')m'$ (for any $m'\in\mathcal{M}'$) is the zero morphism: F lifts to a (unique) morphism $\bar F\colon(\mathbb{A}\otimes_{\mathcal{R}}\mathbb{A})^{**}\to\mathcal{M}'$. Observe that $i\circ\bar F=0$ because $i\circ F=0$, then $\bar F_R=0$ because i_R is injective. Finally, $\bar F=0$ because it is determined by $\bar F_R$; and $F=0$. Likewise, $1\cdot m'=m'$, for all $m'\in\mathcal{M}'$.

In conclusion, $\mathcal{M}'$ is a quasi-coherent $\mathbb{A}$-submodule of $\mathcal{M}$. □

Proposition 5.17. *Let $\mathbb{A}\in\mathfrak{F}$ and $\mathbb{B}$ be functors of $\mathcal{R}$-algebras and assume that $\mathbb{B}$ is an $\mathcal{R}$-submodule of a quasi-coherent module $\mathcal{N}$. Then, any morphism of $\mathcal{R}$-algebras $\phi\colon\mathbb{A}\to\mathbb{B}$ uniquely factors through an epimorphism of algebras onto the quasi-coherent algebra associated with* $\operatorname{Im}\phi_R$.

Proof. By Lemma 5.12, the morphism $\phi\colon\mathbb{A}\to\mathbb{B}$ uniquely factors through an epimorphism $\phi'\colon\mathbb{A}\to\mathcal{B}'$, where $B':=\operatorname{Im}\phi_R$. Obviously B' is a R-subalgebra of $\mathbb{B}(R)$. We have to check that ϕ' is a morphism of functors of algebras.

Observe that if a morphism $f\colon\mathbb{A}\otimes\mathbb{A}\to\mathcal{N}$ factors through an epimorphism onto a quasi-coherent submodule $\mathcal{N}'$ of $\mathcal{N}$ then uniquely factors through $\mathcal{N}'$, because f and any morphism on $\mathcal{N}'$ uniquely factors through $(\mathbb{A}\otimes\mathbb{A})^{**}\in\mathfrak{F}$.

Consider the diagram

$$\begin{tikzcd}
\mathbb{A}\otimes\mathbb{A} \arrow[r, "\phi'\otimes\phi'"] \arrow[d, "m_{\mathbb{A}}"] & \mathcal{B}'\otimes\mathcal{B}' \arrow[r, "i\otimes i"] \arrow[d, "m_{\mathcal{B}'}"] & \mathbb{B}\otimes\mathbb{B} \arrow[d, "m_{\mathbb{B}}"] & \\
\mathbb{A} \arrow[r, "\phi'"] & \mathcal{B}' \arrow[r, "i"] & \mathbb{B} \arrow[r, hook] & \mathcal{N},
\end{tikzcd}$$

where $m_{\mathbb{A}},m_{\mathcal{B}'}$ and $m_{\mathbb{B}}$ are the multiplication morphisms and i is the morphism induced by the morphism $B'\to\mathbb{B}(R)$. We know $m_{\mathbb{B}}\circ(i\otimes i)\circ(\phi'\otimes\phi')=i\circ\phi'\circ m_{\mathbb{A}}$. The morphism $m_{\mathbb{B}}\circ(i\otimes i)\circ(\phi'\otimes\phi')$ uniquely factors onto $\mathcal{B}'$, more concretely, through $m_{\mathcal{B}'}\circ(\phi'\otimes\phi')$. The morphism $i\circ\phi'\circ m_{\mathbb{A}}$ uniquely factors onto $\mathcal{B}'$, effectively, through $\phi'\circ m_{\mathbb{A}}$. Then, $m_{\mathcal{B}'}\circ(\phi'\otimes\phi')=\phi'\circ m_{\mathbb{A}}$ and ϕ' is a morphism of $\mathcal{R}$-algebras. □

Definition 5.18. *We will say that a functor of $\mathcal{R}$-algebras is a functor of proquasi-coherent algebras if it is the inverse limit of its quasi-coherent algebra quotients.*

Examples 5.19. *Quasi-coherent algebras are proquasi-coherent.*

Let $R=K$ be a field, A be a commutative K-algebra and $I\subseteq A$ be an ideal. Then, $\mathbb{B}=\lim\limits_{\substack{\leftarrow\\ n\in\mathbb{N}}}\mathcal{A}/\mathcal{I}^n\in\mathfrak{F}$ (by 5.10) and it is a proquasi-coherent algebra: $\mathbb{B}\simeq\prod_n\mathcal{I}^n/\mathcal{I}^{n+1}$. Then, $\mathbb{B}^*=\oplus_n(\mathcal{I}^n/\mathcal{I}^{n+1})^*=\lim\limits_{\substack{\rightarrow\\ n}}(\mathcal{A}/\mathcal{I}^n)^*$. Therefore, $\mathbb{B}^*$ is equal to the direct limit of the dual of the quasi-coherent algebra quotients of $\mathbb{B}$. Dually, $\mathbb{B}$ is a proquasi-coherent algebra.

Proposition 5.20. *Let $\mathcal{C}^*\in\mathfrak{F}$ be a functor of $\mathcal{R}$-algebras (i.e., a scheme of $\mathcal{R}$-algebras). Then, $\mathcal{C}^*$ is a functor of proquasi-coherent algebras.*

Proof. 1. If $\mathcal{M}^* \in \mathfrak{F}$ and $f\colon \mathcal{M}^* \to \mathcal{N}$ is a morphism of functors of $\mathcal{R}$-modules, then $N' := \operatorname{Im} f_R$ is an R-module of finite type: By Lemma 5.12, f factors via an epimorphism $f'\colon \mathcal{M}^* \to \mathcal{N}'$. $\operatorname{Hom}_{\mathcal{R}}(\mathcal{M}^*, \mathcal{N}') = M \otimes N'$, then $f' = m_1 \otimes n_1 + \cdots + m_r \otimes n_r$, for some $m_i \in M$ and $n_i \in N'$. Hence, $f(w) = \sum_i w(m_i) \cdot n_i$, for all $w \in \mathcal{M}^*$ and $N' = \langle n_1, \ldots, n_r \rangle$.

2. $\mathcal{C}^*$ is a left and right $\mathcal{C}^*$-module, then $\mathcal{C}$ is a right and left $\mathcal{C}^*$-module. Given $c \in C$, the dual morphism of the morphism $\mathcal{C}^* \to \mathcal{C}$, $w \mapsto w \cdot c$ is the morphism $\mathcal{C}^* \to \mathcal{C}$, $w \mapsto c \cdot w$.

3. C is the direct limit of its R-modules of finite type. Let $N = \langle n_1, \ldots, n_r \rangle \subset C$ be an R-module of finite type and let $f\colon \mathcal{C}^{*r} \to \mathcal{N}$ be defined by $f((w_i)) := \sum_i w_i \cdot n_i$. Then, $N' := C^* \cdot N = \operatorname{Im} f_R$ is an R-module of finite type. By Proposition 5.16, $\mathcal{N}'$ is an $\mathcal{C}^*$-submodule of $\mathcal{C}$. Write $N' = \langle n_1, \ldots, n_s \rangle$. The morphism $\mathbb{E}nd_{\mathcal{R}}(\mathcal{N}') \to \oplus^s \mathcal{N}'$, $g \mapsto (g(n_i))_i$ is injective. By Proposition 5.17, the morphism of functors of $\mathcal{R}$-algebras $\mathcal{C}^* \to \mathbb{E}nd_{\mathcal{R}}(\mathcal{N}')$ $w \mapsto w \cdot$ factors through an epimorphism onto a quasi-coherent algebra, $\mathcal{B}'$. The dual morphism of the composite morphism

$$\begin{tikzcd}
\mathcal{C}^* \arrow[r] & \mathcal{B}' \arrow[r] & \mathbb{E}nd_{\mathcal{R}}(\mathcal{N}') \arrow[r, hook] & \oplus^s \mathcal{N}' \arrow[r] & \oplus^s \mathcal{C} \arrow[r, "\pi_i"] & \mathcal{C} \\
w \arrow[rrrrr, mapsto] & & & & & w \cdot n_i
\end{tikzcd}$$

is $\mathcal{C}^* \to \mathcal{B}'^* \hookrightarrow \mathcal{C}$, $w \mapsto n_i \cdot w$. Hence, $n_i \in \mathcal{B}'^*$. Therefore, $\mathcal{C}$ is equal to the direct limit of the dual functors of the quasi-coherent algebra quotients of $\mathcal{C}^*$. Dually, $\mathcal{C}^*$ is a functor of proquasi-coherent algebras.

□

Lemma 5.21. *For any* $\mathbb{M}_1, \ldots, \mathbb{M}_n \in \mathfrak{F}$*, the natural morphism* $(\mathbb{M}_1 \otimes \cdots \otimes \mathbb{M}_n)^{**} \to (\mathbb{M}_1^* \otimes \cdots \otimes \mathbb{M}_n^*)^*$ *is injective.*

Proof. Let us follow the notations $\oplus_{J_i} \mathcal{R} \subseteq \mathbb{M}_i \subseteq \prod_{J_i} \mathcal{R}$. By Lemma 5.1, $\oplus_{J_i} \mathcal{R} \subseteq \mathbb{M}_i^* \subseteq \prod_{J_i} \mathcal{R}$. By induction hypothesis, $\oplus_{J_1 \times \cdots \times J_{n-1}} \mathcal{R} \subseteq (\mathbb{M}_1 \otimes \cdots \otimes \mathbb{M}_{n-1})^* \subseteq \prod_{J_1 \times \cdots \times J_{n-1}} \mathcal{R}$. Since

$$(\mathbb{M}_1 \otimes \cdots \otimes \mathbb{M}_n)^* = \mathbb{H}om_{\mathcal{R}}(\mathbb{M}_1 \otimes \cdots \otimes \mathbb{M}_{n-1}, \mathbb{M}_n^*) = \mathbb{H}om_{\mathcal{R}}((\mathbb{M}_1 \otimes \cdots \otimes \mathbb{M}_{n-1})^{**}, \mathbb{M}_n^*),$$

by Equation 2, $\oplus_{J_1 \times \cdots \times J_n} \mathcal{R} \subseteq (\mathbb{M}_1 \otimes \cdots \otimes \mathbb{M}_n)^* \subseteq \prod_{J_1 \times \cdots \times J_n} \mathcal{R}$. Hence, firstly $(\mathbb{M}_1 \otimes \cdots \otimes \mathbb{M}_n)^{**} \subseteq \prod_{J_1 \times \cdots \times J_n} \mathcal{R}$, by Lemma 5.1, secondly $(\mathbb{M}_1^* \otimes \cdots \otimes \mathbb{M}_n^*)^* \subseteq \prod_{J_1 \times \cdots \times J_n} \mathcal{R}$.

As a consequence, the natural morphism $(\mathbb{M}_1 \otimes \cdots \otimes \mathbb{M}_n)^{**} \to (\mathbb{M}_1^* \otimes \cdots \otimes \mathbb{M}_n^*)^*$ is injective.

□

Theorem 5.22. *Let* $\mathbb{A}, \mathbb{B} \in \mathfrak{F}$ *be two functors of proquasi-coherent algebras. Then,* $(\mathbb{A}^* \otimes \mathbb{B}^*)^* \in \mathfrak{F}$ *is a functor of proquasi-coherent algebras and it holds*

$$\operatorname{Hom}_{\mathcal{R}-alg}(\mathbb{A} \otimes \mathbb{B}, \mathbb{C}) = \operatorname{Hom}_{\mathcal{R}-alg}((\mathbb{A}^* \otimes \mathbb{B}^*)^*, \mathbb{C})$$

for every functor of proquasi-coherent algebras $\mathbb{C}$.

Proof. Write $\mathbb{A} = \lim\limits_{\substack{\leftarrow \\ i}} \mathcal{A}_i$ and $\mathbb{B} = \lim\limits_{\substack{\leftarrow \\ j}} \mathcal{B}_j$. Observe that

$$\begin{aligned}(\mathbb{A}^* \otimes \mathbb{B}^*)^* &= \mathbb{H}om_{\mathcal{R}}(\mathbb{A}^*, \mathbb{B}) \overset{2.15}{=} \mathbb{H}om_{\mathcal{R}}(\lim_{\substack{\rightarrow \\ i}} \mathcal{A}_i^*, \mathbb{B}) = \mathbb{H}om_{\mathcal{R}}(\lim_{\substack{\rightarrow \\ i}} \mathcal{A}_i^*, \lim_{\substack{\leftarrow \\ j}} \mathcal{B}_j) \\ &= \lim_{\substack{\leftarrow \\ i,j}} \mathbb{H}om_{\mathcal{R}}(\mathcal{A}_i^*, \mathcal{B}_j) = \lim_{\substack{\leftarrow \\ i,j}} (\mathcal{A}_i \otimes \mathcal{B}_j)\end{aligned}$$

Then, $(\mathbb{A}^* \otimes \mathbb{B}^*)^*$ is a functor of algebras and the natural morphism $\mathbb{A} \otimes \mathbb{B} \to (\mathbb{A}^* \otimes \mathbb{B}^*)^*$ is a morphism of functors of algebras.

Given a morphism of functor of $\mathcal{R}$-algebras $\phi\colon \mathbb{A} \otimes \mathbb{B} \to \mathcal{C}$, let $\phi_1 = \phi_{|\mathbb{A}\otimes 1}$ and $\phi_2 = \phi_{|1\otimes\mathbb{B}}$. Then, ϕ_1 factors through an epimorphism onto a quasi-coherent algebra quotient $\mathcal{A}_i$ of $\mathbb{A}$, and ϕ_2 factors through an epimorphism onto a quasi-coherent algebra quotient $\mathcal{B}_j$ of $\mathbb{B}$. Then, ϕ factors through $\mathcal{A}_i \otimes \mathcal{B}_j$, and ϕ factors through $(\mathbb{A}^* \otimes \mathbb{B}^*)^*$. Then,

$$\mathrm{Hom}_{\mathcal{R}-alg}((\mathbb{A}^* \otimes \mathbb{B}^*)^*, \mathcal{C}) \to \mathrm{Hom}_{\mathcal{R}-alg}(\mathbb{A} \otimes \mathbb{B}, \mathcal{C})$$

is surjective. It is also injective, because

$$\begin{aligned}\mathrm{Hom}_{\mathcal{R}}((\mathbb{A}^* \otimes \mathbb{B}^*)^*, \mathcal{C}) &= \mathrm{Hom}_{\mathcal{R}}(\mathcal{C}^*, (\mathbb{A}^* \otimes \mathbb{B}^*)^{**}) \\ &\overset{5.21}{\subseteq} \mathrm{Hom}_{\mathcal{R}}(\mathcal{C}^*, (\mathbb{A} \otimes \mathbb{B})^*) = \mathrm{Hom}_{\mathcal{R}}(\mathbb{A} \otimes \mathbb{B}, \mathcal{C})\end{aligned}$$

Then, $\mathrm{Hom}_{\mathcal{R}-alg}(\mathbb{A} \otimes \mathbb{B}, \mathbb{C}) = \mathrm{Hom}_{\mathcal{R}-alg}((\mathbb{A}^* \otimes \mathbb{B}^*)^*, \mathbb{C})$ for every proquasi-coherent algebra $\mathbb{C}$.

A morphism of functors of algebras $f\colon (\mathbb{A}^* \otimes \mathbb{B}^*)^* \to \mathcal{C}$ factors through some $\mathcal{A}_i \otimes \mathcal{B}_j$ because $f_{|\mathbb{A}\otimes\mathbb{B}}$ factors through some $\mathcal{A}_i \otimes \mathcal{B}_j$. Then, the inverse limit of the quasi-coherent algebra quotients of $(\mathbb{A}^* \otimes \mathbb{B}^*)^*$ is equal to $\lim\limits_{\substack{\leftarrow \\ i,j}} (\mathcal{A}_i \otimes \mathcal{B}_j) = (\mathbb{A}^* \otimes \mathbb{B}^*)^*$, that is, $(\mathbb{A}^* \otimes \mathbb{B}^*)^*$ is a proquasi-coherent algebra.

□

Notation 5.23. *Let $\mathbb{A}_1, \ldots, \mathbb{A}_n \in \mathfrak{F}$ be functors of proquasi-coherent $\mathcal{R}$-algebras. We denote $\mathbb{A}_1 \tilde{\otimes} \cdots \tilde{\otimes} \mathbb{A}_n := (\mathbb{A}_1^* \otimes \cdots \otimes \mathbb{A}_n^*)^*$, which is the closure of functors of proquasi-coherent algebras of $\mathbb{A}_1 \otimes \cdots \otimes \mathbb{A}_n$.*

Observe that $\mathbb{A}_1 \tilde{\otimes} (\mathbb{A}_2 \tilde{\otimes} \cdots \tilde{\otimes} \mathbb{A}_n) = \mathbb{A}_1 \tilde{\otimes} \cdots \tilde{\otimes} \mathbb{A}_n$.

Definition 5.24. *A functor $\mathbb{B} \in \mathfrak{F}$ is said to be a functor of proquasi-coherent bialgebras when $\mathbb{B}$ and $\mathbb{B}^*$ are functors of proquasi-coherent $\mathcal{R}$-algebras such that the dual morphisms of the multiplication morphism $m\colon \mathbb{B}^* \otimes \mathbb{B}^* \to \mathbb{B}^*$ and the unit morphism $u\colon \mathcal{R} \to \mathbb{B}^*$ are morphisms of functors of $\mathcal{R}$-algebras.*

Let $\mathbb{B}, \mathbb{B}'$ be two functors of proquasi-coherent bialgebras. We will say that a morphism of $\mathcal{R}$-modules, $f\colon \mathbb{B} \to \mathbb{B}'$ is a morphism of functors of bialgebras if f and $f^*\colon \mathbb{B}'^* \to \mathbb{B}^*$ are morphisms of functors of $\mathcal{R}$-algebras.

Note 5.25. *Let $\mathbb{B} \in \mathfrak{F}$ be a functor of proquasi-coherent algebras. Defining a multiplication morphism $\mathbb{B}^* \otimes \mathbb{B}^* \to \mathbb{B}^*$ (associative and with a unit) is equivalent to defining a comultiplication morphism $\mathbb{B} \to \mathbb{B} \tilde{\otimes} \mathbb{B}$ (coassociative and with a counit), because*

$$\mathrm{Hom}_{\mathcal{R}}(\mathbb{B}^* \otimes \overset{n}{\cdots} \otimes \mathbb{B}^*, \mathbb{B}^*) = \mathrm{Hom}_{\mathcal{R}}(\mathbb{B}, (\mathbb{B}^* \otimes \overset{n}{\cdots} \otimes \mathbb{B}^*)^*) = \mathrm{Hom}_{\mathcal{R}}(\mathbb{B}, \mathbb{B} \tilde{\otimes} \overset{n}{\cdots} \tilde{\otimes} \mathbb{B})$$

In the literature, an R-algebra A is said to be a bialgebra if it is a coalgebra (with counit) and the comultiplication $c\colon A \to A \otimes_R A$ and the counit $e\colon A \to R$ are morphisms of R-algebras.

Proposition 5.26. *The functors $A \rightsquigarrow \mathcal{A}$ and $\mathcal{A} \rightsquigarrow \mathcal{A}(R)$ establish an equivalence between the category of R-bialgebras and the category of functors of $\mathcal{R}$-proquasi-coherent bialgebras (A and $\mathcal{A}(R)$ are assumed to be free R-modules).*

Proof. Observe that if $\mathcal{A}^*$ is a functor of $\mathcal{R}$-algebras then it is a proquasi-coherent algebra, by Proposition 5.20. □

Theorem 5.27. *Let $\mathcal{C}_{\mathfrak{F}-Bialg.}$ be the category of functors $\mathbb{B} \in \mathfrak{F}$ of proquasi-coherent bialgebras. The functor $\mathcal{C}_{\mathfrak{F}-Bialg.} \rightsquigarrow \mathcal{C}_{\mathfrak{F}-Bialg.}$, $\mathbb{B} \rightsquigarrow \mathbb{B}^*$ is a categorical anti-equivalence.*

Proof. Let $\{\mathbb{B}, m, u; \mathbb{B}^*, m', u'\}$ be a functor of bialgebras. Let us only check that $m^*\colon \mathbb{B}^* \to (\mathbb{B} \otimes \mathbb{B})^* = \mathbb{B}^* \tilde{\otimes} \mathbb{B}^*$ is a morphism of functors of algebras. By hypothesis, ${m'}^*\colon \mathbb{B} \to (\mathbb{B}^* \otimes \mathbb{B}^*)^* = \mathbb{B} \tilde{\otimes} \mathbb{B}$ is a morphism of functors of algebras. We have the commutative square:

$$\begin{tikzcd}
\mathbb{B} \arrow[r, "{m'}^*"] & \mathbb{B}\tilde{\otimes}\mathbb{B} \\
\mathbb{B} \otimes \mathbb{B} \arrow[u, "m"] \arrow[r, "{m'}^*_{13} \otimes {m'}^*_{24}"] & \mathbb{B}\tilde{\otimes}\mathbb{B}\tilde{\otimes}\mathbb{B}\tilde{\otimes}\mathbb{B} \arrow[u, "m \otimes m"']
\end{tikzcd}$$

where $({m'}^*_{13} \otimes {m'}^*_{24})(b_1 \otimes b_2) := \sigma({m'}^*(b_1) \otimes {m'}^*(b_2))$ and $\sigma(b_1 \otimes b_2 \otimes b_3 \otimes b_4) := b_1 \otimes b_3 \otimes b_2 \otimes b_4$, for all $b_i \in \mathbb{B}$. Taking duals, we obtain the commutative diagram:

$$\begin{tikzcd}
\mathbb{B}^* \arrow[d, "m^*"'] & & (\mathbb{B}^* \otimes \mathbb{B}^*)^{**} \arrow[ll] \arrow[d, "(m\otimes m)^*"] & \mathbb{B}^* \otimes \mathbb{B}^* \arrow[l, dashed] \arrow[lll, dashed, bend right, "m'"'] \arrow[ddll, dashed, bend left, "m^* \otimes m^*"] \\
\mathbb{B}^*\tilde{\otimes}\mathbb{B}^* & & (\mathbb{B}^* \otimes \mathbb{B}^* \otimes \mathbb{B}^* \otimes \mathbb{B}^*)^{**} \arrow[ll] \arrow[dl, dashed] & \\
& \mathbb{B}^*\tilde{\otimes}\mathbb{B}^*\tilde{\otimes}\mathbb{B}^*\tilde{\otimes}\mathbb{B}^* \arrow[ul, dashed, "m'_{13} \otimes m'_{24}"'] & &
\end{tikzcd}$$

which says that m^* is a morphism of functors of $\mathcal{R}$-algebras.

□

In [6, Ch. I, §2, 13], Dieudonné proves the anti-equivalence between the category of commutative K-bialgebras and the category of linearly compact cocommutative K-bialgebras (where K is a field).

Notation 5.28. *Let $\mathbb{A}$ be a reflexive functor of $\mathcal{K}$-algebras and let $\{\mathcal{A}_i\}$ the set of quasi-coherent quotients of $\mathbb{A}$ such that $\dim_K A_i < \infty$. We denote $\bar{\mathbb{A}} := \lim\limits_{\substack{\leftarrow \\ i}} \mathcal{A}_i$ which is an algebra scheme because $\mathcal{A}_i^*$ is quasi-coherent and $\lim\limits_{\substack{\leftarrow \\ i}} \mathcal{A}_i = (\lim\limits_{\substack{\rightarrow \\ i}} \mathcal{A}_i^*)^*$.*

Proposition 5.29. [2, 5.9] *Let $\mathbb{A}$ be a reflexive functor of $\mathcal{K}$-algebras. Then,*

$$\mathrm{Hom}_{\mathcal{K}-alg}(\mathbb{A},\mathcal{C}^*) = \mathrm{Hom}_{\mathcal{K}-alg}(\bar{\mathbb{A}},\mathcal{C}^*)$$

for all algebra schemes $\mathcal{C}^*$.

Theorem 5.30. *Let $\mathbb{B} \in \mathfrak{F}$ be a functor of proquasi-coherent $\mathcal{K}$-bialgebras. Then, $\bar{\mathbb{B}}$ is a scheme of bialgebras and*

$$\mathrm{Hom}_{\mathcal{K}-bialg}(\mathbb{B},\mathcal{C}^*) = \mathrm{Hom}_{\mathcal{K}-bialg}(\bar{\mathbb{B}},\mathcal{C}^*)$$

for all bialgebra schemes $\mathcal{C}^*$.

Proof. Given any $\mathbb{A}_1,\dots,\mathbb{A}_n \in \mathfrak{F}$ proquasi-coherent algebras then $\overline{\mathbb{A}_1\tilde{\otimes}\dots\tilde{\otimes}\mathbb{A}_n} = \bar{\mathbb{A}}_1\tilde{\otimes}\dots\tilde{\otimes}\bar{\mathbb{A}}_n$, by Proposition 5.29. Then, the comultiplication morphism $\mathbb{B}\to\mathbb{B}\tilde{\otimes}\mathbb{B}$ defines a comultiplication morphism $\bar{\mathbb{B}}\to\bar{\mathbb{B}}\tilde{\otimes}\bar{\mathbb{B}}$, and $\bar{\mathbb{B}}$ is a scheme of bialgebras.

Given a morphism of functors of bialgebras $f\colon \mathbb{B}\to\mathcal{C}^*$, that is, a morphism of functors of algebras such that the diagram

$$\begin{tikzcd} \mathbb{B} \arrow[r] \arrow[d, "f"] & \mathbb{B}\tilde{\otimes}\mathbb{B} \arrow[d, "f\otimes f"] \\ \mathcal{C}^* \arrow[r] & \mathcal{C}^*\tilde{\otimes}\mathcal{C}^* \end{tikzcd}$$

is commutative, the induced morphism of functors algebras $\bar{\mathbb{B}}\to\mathcal{C}^*$ is a morphism of functors of bialgebras. Reciprocally, given a morphism of bialgebras $\bar{\mathbb{B}}\to\mathcal{C}^*$, the composition morphism $\mathbb{B}\to\bar{\mathbb{B}}\to\mathcal{C}^*$ is a mosphism of functors of bialgebras.

□

Corollary 5.31. *Let A be a K-bialgebra and let $\mathcal{B}^*$ be a bialgebra scheme. Then,*

$$\mathrm{Hom}_{\mathcal{K}-bialg}(\bar{\mathcal{A}},\mathcal{B}^*) = \mathrm{Hom}_{\mathcal{K}-bialg}(\bar{\mathcal{B}},\mathcal{A}^*)$$

Proof. It holds that $\mathrm{Hom}_{\mathcal{K}-bialg}(\bar{\mathcal{A}},\mathcal{B}^*) = \mathrm{Hom}_{\mathcal{K}-bialg}(\mathcal{A},\mathcal{B}^*) = \mathrm{Hom}_{\mathcal{K}-bialg}(\mathcal{B},\mathcal{A}^*)$ $= \mathrm{Hom}_{\mathcal{K}-bialg}(\bar{\mathcal{B}},\mathcal{A}^*)$.

□

Note 5.32. *The bialgebra $A^\circ := \mathrm{Hom}_{\mathcal{R}}(\bar{\mathcal{A}},\mathcal{R})$ is sometimes known as the "dual bialgebra" of A and Corollary 5.31 says (dually) that the functor assigning to each bialgebra its dual bialgebra is autoadjoint (see* [1, 3.5]*).*

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